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Calibration accuracy of three variants of the Libor Market Model

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This paper explores the calibration accuracy of three variants of the Libor Market Model: the Displaced Diffusion Libor Market Model (DD-LMM), and two different extensions, namely the DD-LMM with Constant Elasticity Volatility (DD-LMM-CEV), and the DD-LMM with Stochastic Volatility (DD-SV-LMM).

Our study is motivated by a context of an increasing complexity of risk-neutral valuation models in the insurance industry, combined with a growing regulatory attention on these models. As part of the regulatory guidance, e.g., as prescribed in Solvency II, there exists a strong requirement on the ability of the interest rate model to replicate observed market prices, which has been one key driver for the increase of model complexity in the insurance practice, especially in terms of number of parameters involved.

We perform a numerical study for a set of different dates and currencies by comparing the swaption volatilities extracted from the market and targeted in the calibration process, with the volatilities obtained from the model simulation. Our two main conclusions are as follows:

1. We show that the most parametrised model (DD-SV-LMM) is not always the one leading to the better replication, and that the less parametrised DD-LMM and DD-LMM-CEV models can lead to overall better fits, the latter presenting the best results overall.
2. We demonstrate numerically that this observation can be partly attributed to the better quality of the so-called freezing approximation involved in the pricing formulas for the DD-LMM and the DD-LMM-CEV models, while the presence of a stochastic volatility component in the DD-SV-LMM model can imply larger numerical discrepancies in the pricing approximations in some cases.

This paper will cover the following topics:

- Analysis of market conditions for the three dates of interest.
- Overview of the three variants of the Libor Market Model.
- Comparison of the market-consistent property of the three models.
- Refinement of the DD-LMM-CEV calibration.

Market consistency

A market-consistent value of an asset or liability refers to:

"...its market value, if it is readily traded on a market at the point in time that the valuation is struck, and, for any other asset or liability, a reasoned best estimate of what its market value would have been had it been readily traded at the relevant valuation point."[1]

Market-consistent economic scenarios are at the core of the measurement of the Technical Provisions under Solvency II. European Insurance and Occupational Pensions Authority (EIOPA) guidelines[2] states that:

"The calculation of technical provisions shall make use of and be consistent with information provided by the financial markets and generally available data on underwriting risks (market-consistency)."

In practice, market consistency is a subtle concept, as the perfect replication of all financial market data, including in particular prices, implied volatilities (IV) etc., is not possible.

From a theoretical point of view, assuming standard arbitrage-free and complete market rules, implies that all market prices can be determined under a unique arbitrage-free "risk-neutral" probability measure.[3] Under such a framework, "risk-neutral" models can be calibrated so as to replicate financial market prices. Nevertheless, in reality, different sets of options can refer to different risk-neutral measures, and no financial model can perfectly depict such complexity. In addition, given the

[1] Kemp (2009). Market Consistency: Model Calibration in Imperfect Markets.

[2] Article 6 from Directive Solvency.

[3] See, e.g., El Karoui et al. (2016) Market inconsistencies of the market-consistent European life insurance economic valuations: pitfalls and practical solutions. For more insights, one can refer to Becherer & Davis (2008). Arrow–Debreu prices.

wide range and variety of quoted options, it is impossible that any financial model captures all prices and characteristics with a unique set of parameters. Insurers must decide the options (type, maturities, tenors, moneyness) they want to embed in the calibration process, along with the weights to be applied to each of them in the optimisation problem. This choice must be well justified,[4] and must be relevant with the specificities of the insurance products.

On the other hand, another particularity of market consistency for insurance technical provisions valuation lies in the very basics of market valuation: the risk-free rate (RFR) per maturity as provided by EIOPA:[5]

"The risk-free interest rate term structure […] underpins the calculation of liabilities by insurance and reinsurance undertakings. EIOPA is required to publish the risk-free interest rate."

The regulatory curve is slightly different from the traditional market swap rates (see the previous technical documentation[6]) and it impacts the ability to assess market prices when using usual options valuation closed-form formulas. Indeed, these prices are functions of RFR and IV. As a consequence, when the RFR differs from the market reference, it is impossible to reconcile both option prices and IV as provided by financial markets. To this extent, insurers generally (1) use the regulatory RFR, (2) consider market IV invariant to the curve, or (3) convert volatility quotations into "pseudo-prices" with the Black or Bachelier formulas. These prices then constitute the model's calibration targets.

The rest of this paper is based on the process described in the previous paragraph. The comparison of rate models is carried out by analysing the replication quality of swaption volatilities, which are in practice the preferred calibration tool for insurers.

Market environment

In the context of this paper, the study is carried out based on at-the-money (ATM) and away-from-the-money (AFM) swaptions at three dates, 31 December 2019, 31 March 2020 and 31 December 2020. For AFM swaptions the 10-year tenor is considered because it is generally the most traded tenor and as such it constitutes a reference in the insurance industry.

We plot the market volatility in basis points (bps) against the strike for the three different dates: 31 December 2019, 31 March 2020 and 31 December 2020 for different maturities and tenor 10.

FIGURE 1: MARKET SKEW AT MATURITY 4 AND TENOR 10

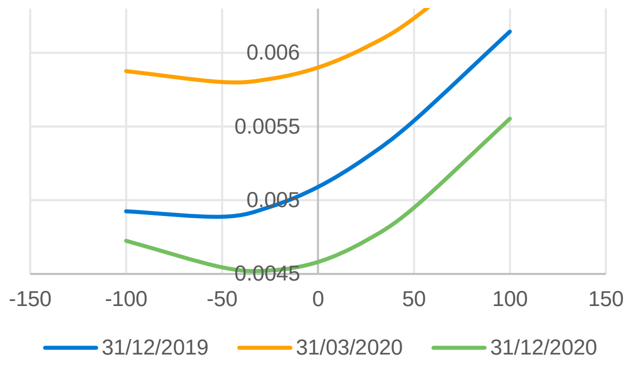


FIGURE 2: MARKET SKEW AT MATURITY 10 AND TENOR 10

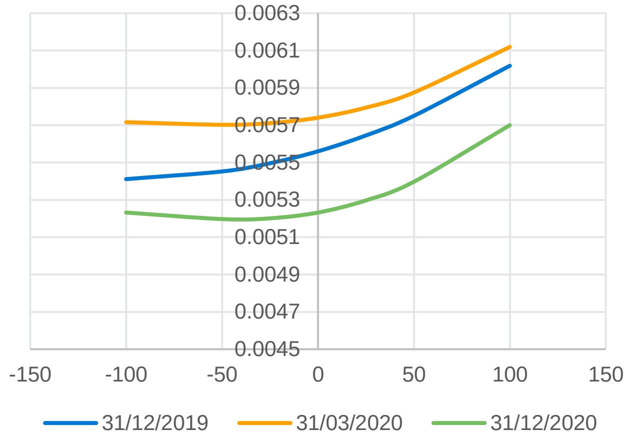
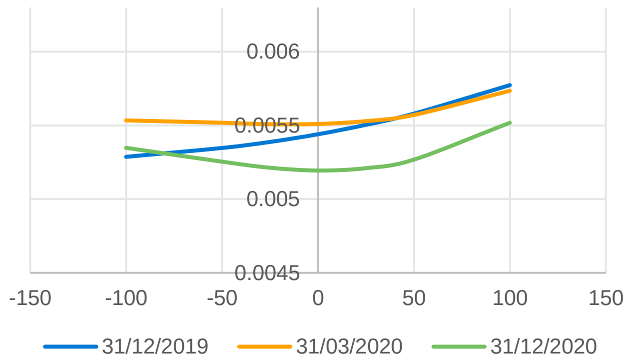


FIGURE 3: MARKET SKEW AT MATURITY 15 AND TENOR 10



[4] For more information, see the Milliman White Paper "Neural Network Calibration of the DDSVLMM Interest Rates Model, and Application to Weights Calculation," available at https://ie.milliman.com/en-gb/insight/neural-network-calibration-of-the-ddsvlmm-interest-rates-model.

[5] EIOPA BoS-19/408(09/19).

[6] Ibid.

The comparison of the swaption implied volatilities between the three dates leads to the following observations:

- At 31 December 2019, we observe a flattening of the curve, with a rather strong slope. This flattening represents a challenge for the DD-LMM-CEV, which is discussed in the following sections.
- At 31 March 2020, we observe high volatility values, with a smile on most maturities. This can be explained by the COVID-19 crisis at the beginning of 2020, followed by a spike in the implied volatilities.
- At 31 December 2020, we observe a smile for all maturities, more pronounced than at 31 March 2020, but with lower values. Compared to 31 March 2020, the average reduction in the volatilities is 9.4 bps.

One relevant criterion to gauge the market consistency of a model is its ability to replicate the volatility skew and smiles.

The DD-SV-LMM can accurately replicate volatility smiles by construction thanks to a stochastic volatility process (see part modelling interest rates). Nevertheless, the Monte Carlo replication of this model might encounter difficulties in replicating certain market conditions due to the freezing approximations embedded in its calibration. On the other hand, the DD-LMM and the DD-LMM-CEV both rely on deterministic volatilities processes. As such, they cannot reproduce complex smile profiles. However, they allow us to capture skew phenomena thanks to a shift and an elasticity parameter (DD-LMM-CEV only). Further technical details about these models, are presented in the next section.

Libor Market Models

Over the years, the financial literature has introduced various classes of risk-neutral interest rate models, among which are the very popular short-rate models. The early 2000s witnessed the development of the so-called Libor market models based on observable forward rates as opposed to the non-observable short rate. Nowadays, these models are widely used by insurers, in particular because they provide a good replication capability of the market prices. We present in this section the three variants of the Libor market model, namely: the Libor market model with a displacement coefficient (DD-LMM), the Libor market model with a displacement coefficient and elasticity coefficient (DD-LMM-CEV) and finally the LIBOR market model with a displacement coefficient and stochastic volatility (DD-SV-LMM). These three models are available in the Milliman CHESS™ economic scenario generator.[7]

Let us denote $\tilde{F} = F_j(t) + \delta$ the shifted forward rate where $F_j(t)$ defines the forward rate between two consecutive dates and δ the displacement coefficient (shift), allowing the modelling of negative rates and capturing skew behaviours. The DD-LMM-CEV introduces an elasticity parameter γ to improve skew replication while the DD-SV-LMM encompasses stochastic volatility following a Cox-Ingersoll-Ross (CIR) dynamics and allowing smile replication.

The general form of the dynamics of the shifted forward rates $\tilde{F}_k\,(t)$ under the k+1 forward neutral measure is, for the three models:

$$d\tilde{F}_k(t) = \tilde{F}_k(t)^{\gamma} \times \sum_{q=1}^{N_f} \zeta_k^q(t)\, dZ_{k+1}^q\,(t)$$

with $N_f \geq 1$ being an integer equal to the number of factors in the model, $(Z^q)_{q\,\in[\![1,N_f]\!]}$ a multidimensional Brownian motion under the K+1 forward neutral measure and $(\zeta_k^q)_{q\,\in[\![1,N_f]\!]}$ the volatility pattern.

FIGURE 4: LIBOR MARKET MODELS

	DD-LMM	DD-LMM-CEV	DD-SV-LMM
Elasticity	$\gamma = 1$	$\gamma > 0$	$\gamma = 1$
Volatility function	$\zeta_k^q(t)$ is a deterministic function	$\zeta_k^q(t)$ is a deterministic function	$\zeta_k^q(t)$ is a stochastic function, driven by a CIR process
Number of parameters	7 parameters	8 parameters	9 parameters

In their original form, the models are not tractable, in particular as they do not belong to the class of affine processes. Hence, in order to recover classical pricing formulas, an approximation called “freezing” is considered. Freezing consists in approximating the swap rates dynamics by fixing to their initial value some ratios of forward and swap rates, involved in the drift of swap rates as well as in the drift of the stochastic variance of the DD-SV-LMM. As highlighted by Jaeckel and Rebonato (2002)[8] and Rebonato (2002),[9] the validity of the freezing approximations lies in frozen quantities having a small volatility, and their expectations being centered around the frozen values. As such, this approximation may be less accurate when more stochastic sources are embedded in the model, such as when a stochastic volatility is captured. Once closed-form pricing formulas have been derived, the calibration of the parameters is then performed by minimising the distance between the market swaption prices and the prices obtained from the pricing approximations (so-called model prices).

[7] More information is available at https://www.milliman.com/en/products/milliman-chess.

[8] Jaeckel, P. and Rebonato, R. (2002). The link between caplet and swaption volatilities in a BGM/J framework: Approximate solutions and empirical evidence.

[9] Rebonato, R. (2002). Modern Pricing of Interest-Rate Derivatives: The LIBOR Market Model and Beyond.

Calibration and results

In order to compare the market-consistency properties of the DD-LMM, DD-LMM-CEV and DD-SV-LMM models at 31 December 2019, 31 March 2020 and 31 December 2020, we break down the analysis into four steps:

- **Step 1**: Comparison of the repricing errors between market and Monte Carlo volatilities assessed by simulating models and relying on a Monte Carlo pricing of the swaptions.
- **Step 2**: Analysis of the skew/smile replication.
- **Step 3**: Analysis of the freezing error induced by the approximations embedded in the calibration process of the models.
- **Step 4**: Analysis of the calibrated parameters.

Overall comparison of ATM and AFM calibrations

For each date, we calibrate the DD-LMM, DD-LMM-CEV and DD-SV-LMM on the market swaption volatilities. We then simulate the models and compute swaption volatilities implied from Monte Carlo valuation techniques applied to the simulated payoffs. In the following, the corresponding volatilities are referred as Monte Carlo volatilities; they are compared to the market volatilities by calculating the average of the absolute differences.

The table in Figure 5 shows the average absolute errors between at-the-money (ATM) market and Monte Carlo volatilities:

FIGURE 5: AVERAGE ABSOLUTE ERRORS BETWEEN ATM MONTE CARLO AND MARKET VOLATILITIES

	DATE	DD-LMM	DD-LMM-CEV	DD-SV-LMM
ATM	31/12/2019	0.0200%	0.0197%	0.0192%
	31/03/2020	0.0210%	0.0208%	0.0351%
	31/12/2020	0.0192%	0.0182%	0.0177%

Note: On each date, the following colour codes are used: green for best result, blue for second-best result, orange for worst result.

For two of the three dates of interest, we notice the DD-SV-LMM provides the lower level of calibration error for ATM swaptions, which is followed by the DD-LMM-CEV and DD-LMM, with overall close values of ATM calibration errors. This ranking seems rather intuitive, with the calibration error decreasing with the number of parameters available for calibration. As at 31 March 2020, the DD-LMM-CEV is, however, ranked first, while the DD-SV-LMM provides the highest calibration error at this same date, which appears as counterintuitive.

To complete the analysis, the table in Figure 6 shows at the 10 years tenor the average absolute deviation between away-from-the-money (AFM) market and Monte Carlo swaption volatilities.

FIGURE 6: AVERAGE ABSOLUTE ERRORS BETWEEN AFM MONTE CARLO AND MARKET VOLATILITIES

	DATE	DD-LMM	DD-LMM-CEV	DD-SV-LMM
ATM	31/12/2019	0.0256%	0.0331%	0.0240%
	31/03/2020	0.0233%	0.0192%	0.0374%
	31/12/2020	0.0212%	0.0198%	0.0207%

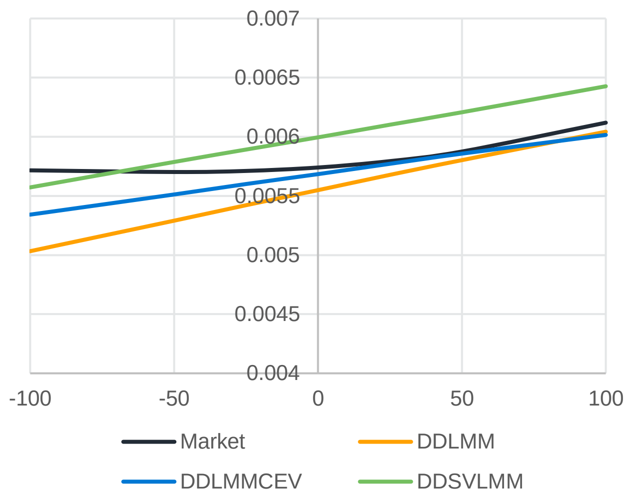
These results confirm the observations above; the DD-SV-LMM exhibits higher calibration errors at 31 March 2020 compared to both the DD-LMM-CEV and DD-LMM. In the following, we will further analyse the calibration at 31 March 2020, which led to relatively high replication errors for the DD-SV-LMM.

Additionally, we note that the results of the DD-LMM-CEV at 31 December 2019 highlight some difficulties to reproduce AFM data. In the last section below we will dig into this issue and provide insights for improvement of the DD-LMM-CEV calibration.

Comparison of skew replication

We now compare the smile/skew replication of the three models. To this extent the graph in Figure 7 illustrates the volatility smile for the swaptions of maturity 10 years and tenor 10 years, at 31 March 2020. Strikes are displayed in basis points (bps) on the x-axis.

FIGURE 7: SMILE OF ATM MONTE CARLO VOLATILITIES FOR MATURITY 10 AND TENOR 10 AT 31/03/2020



Firstly, this plot emphasises that the DD-LMM-CEV and the DD-LMM allow us to capture linear (skew) profiles thanks to the shift and the elasticity parameters but we cannot reproduce the smile convexity because it relies on deterministic volatility processes.

The DD-LMM-CEV succeeds in reproducing the overall shape of the skew, in particular for positive strikes, whereas in comparison the DD-SV-LMM seems to overestimate the volatility smile for those strikes. Because the very nature of the DD-SV-LMM lies in its ability to accurately reproduce complex smile profiles, the following will carefully analyse the performance of the model calibration to demonstrate that this unexpected phenomenon is partly due to the swaption pricing approximations involved in the calibration process.

Analysis of the model volatilities

The calibration of the DD-LMM, DD-LMM-CEV and DD-SV-LMM models relies on pricing approximations to compute swaption volatilities, referred as model volatilities, based on the freezing technique.

In order to understand the deterioration of the DD-SV-LMM Monte Carlo replication at 31 March 2020, the table in Figure 8 compares the average absolute errors between market and model volatilities for both ATM and AFM surfaces.

FIGURE 8: MEAN OF ABSOLUTE ERROR BETWEEN MODEL AND MARKET VOLATILITIES AT 31/03/2020

	DD-LMM	DD-LMM-CEV	DD-SV-LMM
ATM	0.02110%	0.02080%	0.01760%
AFM	0.02280%	0.01950%	0.0199%

The average model error is much smaller for the DD-SV-LMM in comparison to the Monte Carlo errrors presented in the previous sections. This finding is consistent with the fact that the DD-SV-LMM embeds more degrees of freedom (parameters) in the calibration process.

To complete the analysis, we plot the absolute differences between the model and the Monte Carlo ATM volatilities, respectively, for the DD-SV-LMM and the DD-LMM-CEV as a comparison basis, which therefore measure the impact of the pricing approximations involved in the calibration process and the discretisation error (up to simulation error of the Monte Carlo estimate).

FIGURE 9: ATM VOLATILITY ABSOLUTE DIFFERENCE MODEL/MONTE CARLO FOR DD-SV-LMM AT 31/03/2020

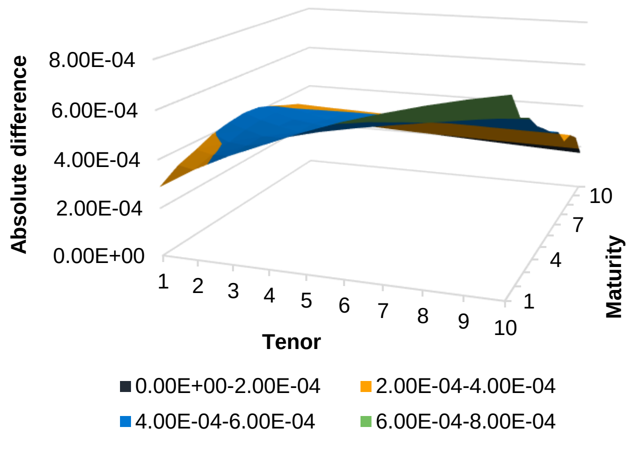


FIGURE 10: ATM VOLATILITY ABSOLUTE DIFFERENCES MODEL/MONTE CARLO FOR DD-LMM-CEV AT 31/03/2020

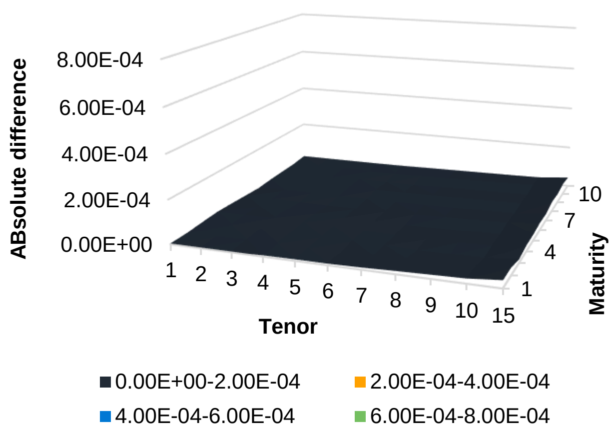
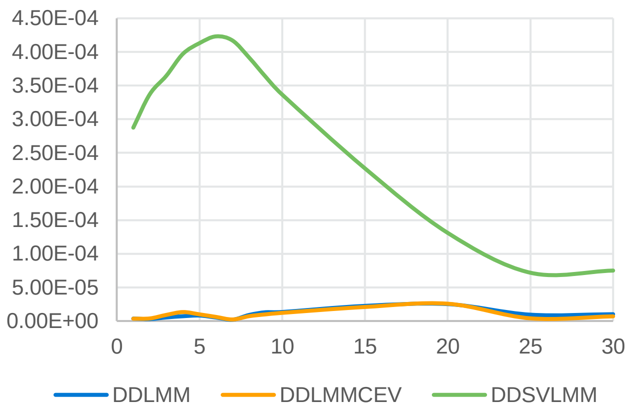


FIGURE 11: CUT OF THE ATM VOLATILITY ABSOLUTE DIFFERENCES MONTE CARLO/MODEL FOR TENOR 1-YEAR AT 31/03/2020



The volatility differences measured for the DD-SV-LMM are significantly higher than the ones of the DD-LMM-CEV.

For the DD-LMM-CEV, we note that the differences are increasing with the tenor. From a theoretical point of view, this could be due to the number of quantities frozen in the swap rates dynamics (required for swaption pricing) increasing with the tenor and implying higher volatility of the frozen quantities. This phenomenon appears more clearly focussing on the 1-year tenor because the swap rates of tenor 1-year are also yearly forward rates and as such their dynamics under the DD-LMM-CEV framework do not use the freezing approximations. This observation is not true for the DD-SV-LMM because an additional freezing occurs in the stochastic volatility dynamics; from this analysis, we can conclude that this is the main source of error, because contrarily to the DD-LMM-CEV, the DD-SV-LMM Monte Carlo volatilities of tenor 1-year significantly differ from model volatilities.

These results highlight that the DD-SV-LMM might encompass a higher freezing error than the DD-LMM and that the errors between market and Monte Carlo volatilities are partly driven by the deterioration of the freezing approximations under the 31 March 2020 stressed market conditions. On the other hand, the less parametrised and deterministic volatility DD-LMM-CEV model provides at this date a lower freezing error combined with a reasonable model volatility fit, so as a result a more robust calibration error involving Monte Carlo volatilities.

Analysis of the calibrated parameters

We discuss below the parameters calibrated by minimisation of the swaption quadratic deviations, for the DD-LMM-CEV (eight parameters) and for the DD-SV-LMM (nine parameters).

The volatility structure $\sigma_k(t)$ is decomposed into two parametric functions. For all $k \in \mathbb{N}^*$ and for all $t \in]T_{k-1}, T_k]$, $\sigma_k(t) = \Phi(t) \times g(T_k - t)$ with g a time-to-maturity-depending function so that, for all $u \in \mathbb{R}^+$, $g(u) = (bu + a) \times e^{-cu} + d$ (Rebonato form).

The parameters a, b, c and d drive the shape of the forward volatility function for all models at a given time.

Φ is a time-depending scaling factor deterministic for the DD-LMM-CEV whereas it is stochastic and ruled by a CIR process for the DD-SV-LMM. Compared to DD-LMM-CEV, the scaling function of the DD-SV-LMM relies on two additional parameters; ϵ, which drives the volatility of volatility, and ρ, which drives the correlation with stochastic volatility. **θ is the asymptotic scaling level and κ is the convergence speed towards θ.**

It is interesting to note that the Rebonato form parameters are relatively close for the two models. At 31 March 2020, we also remark that, for the DD-SV-LMM, the volatility of the volatility ϵ doubles and that the CIR long-term average of the variance diverges more significantly from its initial state ($\Phi(0) = 1$), which implies a greater variability of the stochastic variance process that is likely to increase the skew of the forward distributions as described in the table in Figure 12.

FIGURE 12: CALIBRATED EPSILON OF DD-SV-LMM

	31/12/2019	31/03/2020	31/12/2020
ϵ	24%	49%	22%
κ	13%	60%	5%
θ	32%	26%	43%

Recall that the quality of the freezing approximations lies in frozen quantities having a small volatility, and their expectation being centered around the frozen values. The spike in the variability of the stochastic variance process at 31 March 2020 challenges these two assumptions and deteriorates the robustness of the freezing approximations of the DD-SV-LMM, especially regarding the freezing considered in the stochastic volatility dynamics.

Eventually, these calibrations suggest a relationship between the shift and the elasticity parameters of the DD-LMM-CEV. We will explore this empirical finding in the next section in order to improve the calibration of the model.

Alternative method of calibration for DD-LMM-CEV

In light of the previous DD-LMM-CEV calibration results, we aim at empirically establishing a linear link between the shift and the elasticity. To this extend, for each date (31 December 2019, 31 March 2020 and 31 December 2020), we perform 100 calibrations of the DD-LMM-CEV for different elasticity and shift initialisations in the numerical optimisation routine. The graphs in Figures 13 to 15 present the resulting calibrated elasticity parameters in terms of the calibrated shift parameters.

FIGURE 13: CALIBRATED ELASTICITY (Y-AXIS) IN TERMS OF THE CALIBRATED SHIFT (X-AXIS) AT 31/12/2019

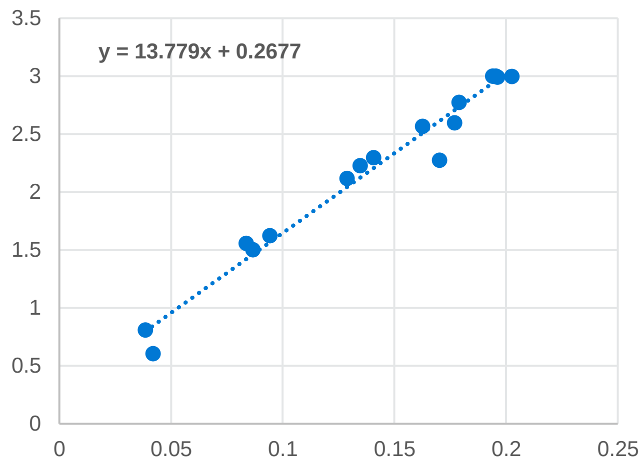


FIGURE 14: CALIBRATED ELASTICITY (Y-AXIS) IN TERMS OF THE CALIBRATED SHIFT (X-AXIS) AT 31/03/2020

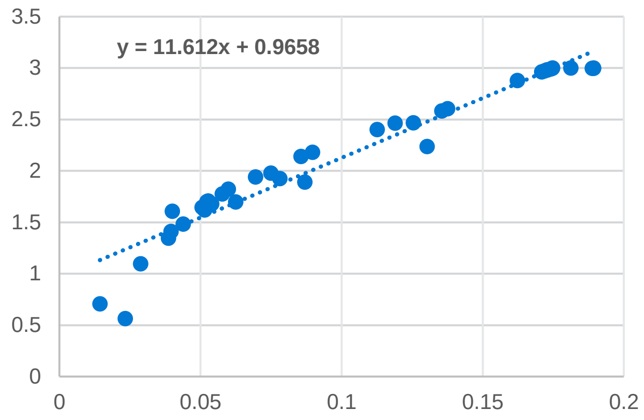
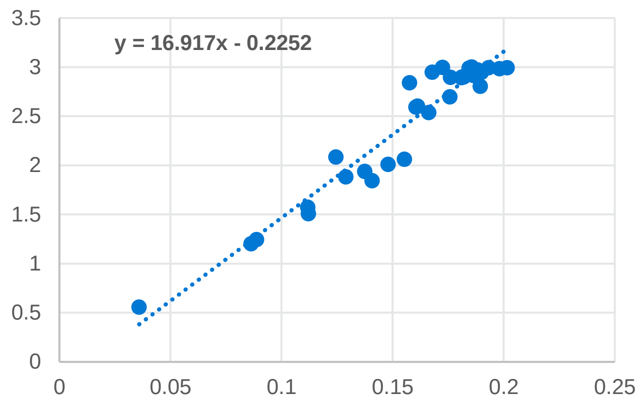


FIGURE 15: CALIBRATED ELASTICITY (Y-AXIS) IN TERMS OF THE CALIBRATED SHIFT (X-AXIS) AT 31/12/2020



We note that there is an approximate linear relationship between the shift and the elasticity as at 31 December 2019, 31 March 2020 and 31 December 2020. For information the regression R^2 are, respectively, 0.9382, 0.9714 and 0.9459.

In order to improve the 31 December 2019 DD-LMM-CEV AFM calibration, we recalibrate the model considering the initial elasticity value (starting point set within the optimisation routine) determined with the linear function estimated at 31 December 2019. The table in Figure 16 compares the average absolute errors between market and Monte Carlo volatilities for both ATM and AFM surfaces.

FIGURE 16: MEAN OF ABSOLUTE ERROR BETWEEN MONTE CARLO AND MARKET VOLATILITIES AT 31/12/2019

	DD-LMM	DD-LMM-CEV	DD-SV-LMM
ATM	0.02000%	0.01990%	0.01920%
AFM	0.02560%	0.02350%	0.024%

We note that the AFM error of the DD-LMM-CEV is greatly reduced (formally 0.0331%), becoming even better than that of the DD-SV-LMM without deteriorating the ATM error. This analysis points out the importance of the elasticity choice for the improvement of the DD-LMM-CEV calibration stability.

Conclusion

To conclude, the most parametrised model (DD-SV-LMM) is not always leading to the better replication in times of stressed market conditions (such as 31 March 2020), as the increase in the volatility is likely to deteriorate the quality of the approximations made for its calibration. On the other hand, this paper highlights the potential of the DD-LMM-CEV model, showing an interesting trade-off between the number of parameters available for calibration and the quality of the pricing approximations involved. Still, it is worth noticing that the elasticity coefficient must be carefully determined in order to recover satisfactory calibration results; as such, this paper has introduced a promising technique to set the starting point of the optimisation procedure for the elasticity parameter as a function of the displacement factor.

If you have any questions or comments on this paper or any other aspect of economic scenario generation, please contact your local Milliman consultants or the contact links below.

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