Tigran Kalberer

Capital allocation for stochastic simulation approaches

Introduction – the relevance of capital allocation

Sound risk management approaches require a full "dashboard" of instruments to measure, monitor and budget risks. There is not one single metric to measure and manage risks, but a full suite of metrics should be at the disposal of the professional risk manager. The metrics range from not to be underestimated qualitative approaches to highly sophisticated quantitative approaches, e.g. the determination of the reaction of the portfolios to different relevant (extreme) scenarios (or sensitivities) or the determination of required capital, as required for solvency-purposes by some regulators.

In this document we will focus on the latter, required capital. This requires the definition of required capital first.

Typically, and loosely speaking, required capital is the amount of capital necessary to ensure that adverse situations do not occur too often in a pre-specified period.

To make this definition operational the vague terms in this definition have to be clearly specified, we will give the typical specifications here, but other definitions of course are possible:

- "capital": amount of assets exceeding the assets covering the market-consistent value of liabilities at t=0
- "adverse situation": market-consistent net value is negative at t=1
- "not too often": only 1 in 200 cases
- "pre-specified period": one year

In (not very helpful) mathematical terms, we need to define the amount RC, such that:
$\text{Prob}(-\Delta NAV_1 \leq RC) = 0.995$, where NAV_1 is a random variable, of course depending on NAV_0, which is $Assets_0$- MVL_0, where MVL_0 is the market value of the liabilities (possibly, but we will not focus on this, including a market value margin) and ΔNAV_1 is just (NAV_1-NAV_0).

Keeping capital in a company comes at a cost. Capital generates frictional costs, e.g. agency costs or asset management costs, but also reduces the LLPO (Limited liability put option) of the shareholder. On the other hand holding capital is beneficial as it allows protecting franchise value and reduces the cost of raising capital at times which are not optimal for such a purpose. There are also tax impacts to be considered. Consumption of capital and beneficial impact however are not necessarily occurring uniformly over all portfolios considered. Thus it is important to measure which portfolio consumes which amount of capital and take this consumption into account when making management decisions.

This however is not as easy as it seems that due to the very nature of insurance business there is typically a large amount of diversification between (the risks of) different portfolios. Thus required capital is typically not additive.

A very popular and theoretically sound approach to allocate capital to different portfolios is the so-called Euler approach, which we define below. Basically this approach allocates capital according to the marginal impact each portfolio has on the total required capital.

So this approach allows to answer most (but not all) questions around capital which have to do with changing the volumes of the portfolios considered, e.g. selling new business in a portfolio which has similar risk characteristics to that portfolio.

Therefore this capital allocation approach is very popular and for good reasons.

It is however, far from trivial to determine the Euler-allocated capital for a sub-portfolio.

The reason for this is that typically the required capital is determined using stochastic simulation.

While this approach allows determining the total required capital relatively easy, the determination of the Euler-allocated capital for a sub-portfolio using the standard simulation approach is numerically unstable due to the large variance of the estimator. Therefore a number of approaches have been developed to overcome this problem. These approaches however are based on general approaches from the academic research area, which do not take the specific setting of our problem into account, but which we can exploit to define a numerically stable, highly convergent, approach to determine the Euler-allocated capital of a sub-portfolio. This approach is superior to the more general typical approaches. But this of course comes at a cost, the approach is only applicable in a specific setting. This however is the setting we have in most insurance-related versions of the problem to determine Euler-allocations. So we just make use of the fact that we know where the approach is applied and utilize the specifics this environment shows. The requirements for this approach are listed below.

In the remainder of the document we will describe the approach and present a case study, assessing the improvement in accuracy using our approach.

The general set-up

A typical stochastic simulation approach for determining the required capital for an insurance company consists of the following steps:

1) The generation of stochastic simulations for all relevant risk factors which are described with a multivariate distribution reflecting their realistic behavior, including possibly complex dependency structures. Generating these simulations is a non-trivial problem and outside of the scope of this document.

2) Functions, which translate each combined realization of all risk factors into their impact on the balance sheet (NAV) of all involved entities (which might be legal entities, separate funds etc.). Again, producing these functions is a non-trivial problem and outside of the scope of this document.

3) Consolidation mechanisms, which combine all these results for insurance groups, preferably reflecting also Capital and Risk Transfer Instruments (CRTIs). It is a complex problem to identify, describe and model these CRTIs.

These steps are by no means easy to design. However, we do not aim to elaborate on these.

Using this aforementioned approach we can estimate the Value at Risk ("VaR") and expected shortfall ("ES") of the portfolio (see below). But we struggle to determine the capital allocation of sub-portfolios to the portfolio. This is the problem which we will address in this paper.

Notation

The corresponding terminology is:

- We assume n relevant risk-factors $X_1,X_2,\ldots,X_{n-1},X_n$ and denote these by the vector $X=(X_1,X_2,\ldots,X_{n-1},X_n)$. Relevant risk factors are all measurable aspects of reality which impact the net value of the portfolios considered, as interest rates in all relevant currencies, claims ratios etc.

- We generate s simulations of these risk factors and obtain therefore s realizations $x^i = (x^i_1,\ldots,x^i_n)$.

- We furthermore assume m portfolios, subject to these risk-factors, and the sum of these portfolios, which is denoted by "parent portfolio". Our aim is to analyze the m portfolios and the "parent portfolio" in terms of capital allocation.

- The negative change in net asset value for these portfolios is described by functions $P^{(1)}(X)$, $P^{(2)}(X),\ldots, P^{(m)}(X)$. This could for instance be polynomials derived using a LSMC-approach. In fact any other approaches producing proxy function could be used here, e.g. curve-fitting or replication portfolios. But using polynomials offers a distinctive advantage as we will see below.

- Clearly the negative net asset value change of the parent portfolio is $P(X):=\sum_{j\le m} P_j(X)$.

- The functions which evaluate a realization $(x^i_1,\ldots,x^i_n)$ have for reasons of notational accuracy, so we define for all j: $p_j(x^i_1,\ldots, x^i_n):=P^{(j)}(X)\ (x^i_1,\ldots,x^i_n)$.

- For convenience we use the following definitions:

  – We split the set X of risk factors into the first n-1 and the last factor, i.e. $X=(Y,Z)$, with $Y=(X_1,X_2,\ldots,X_{n-1})$ and $Z=(X_n)$.

  – In order to analyze the impact of the m-th portfolio, we split the "parent portfolio" into $P^{(m)}(Y,Z)$ and the sum of all remaining portfolios $P^{(j<m)}(Y,Z)=\sum_{j<m} P^{(j)}(Y,Z)$. The "parent portfolio" can hence be written as $P(Y,Z)=P^{(m)}(Y,Z)+P^{(j<m)}(Y,Z)$.

  – Without loss of generality we can assume that the functions P, $P^{(m)}$ and $P^{(j<m)}$ have a finite set of roots w.r.t. Z, for all given realizations of Y.

Determining required capital by stochastic simulation

The value at risk VaRq at a certain quantile q for the "parent portfolio", is typically estimated by choosing VaR_q such that

$$Prob(P(X)\le VaR_q) \approx \tfrac{1}{s}\textstyle\sum_i 1\{p(x^i)\le VaR_q\}=q.$$

Here, 1{statement} is the indicator function, defined as 1 if the statement is true and 0 otherwise. Sorting the results $p(x^i)$, i=1,…,s facilitates the determination of VaR more easily than using this sum. Non-parametric confidence intervals for this estimation of VaR can be determined easily, see [4].

Euler-allocation

A typical question in a real risk management context is:
"What is the contribution of portfolio m to the overall VaR for the parent portfolio?"

We aim to determine the contribution of the portfolio m to the overall VaR, which we denote with C_m. Many approaches have been suggested to determine this contribution. One approach often mentioned in theory and practice is the Euler-allocation:

$$C_m:=\tfrac{\partial}{\partial h}VaR_\alpha(h)\,|_{\{h=0\}} =\tfrac{\partial}{\partial h}VaR\big(P(X)+hP^{(m)}(X)\big)\,|_{\{h=0\}}.$$

While this is the original definition of the Euler-allocation, it is not a well-suited definition in a stochastic-simulation setting. Thus equivalent definitions are typically used. A very popular alternative is derived in [6]: $C_m:=E\big(P^{(m)}(X)\,|\,P(X)=VaR\big)$. While applying this definition we incurred difficulties however because the concept of conditional expectations is less well-defined as one would think, causing real problems.

Note that it is not a straightforward and easy task to determine this risk allocation stochastically. Many approaches have been suggested, but all suffer from numerical instability.

A very naïve approach would be to determine the (nearest) simulation for which p(simulation)=VaR holds and then evaluate P^{m}(simulation).

Why is this not a good approach? This is because we actually have a whole manifold of simulations with p(simulation)=VaR and all what we do is pick exactly one of these simulations arbitrarily out of this manifold, i.e. estimate the quantity $(P^{(m)}(X)|P(X)=VaR)$ by using one simulation! Obviously this estimation has a large confidence interval around C_m and is basically unusable.

The other approaches mostly try to avoid this situation by using simulations near the VaR and averaging these using clever weighting.

The new "main risk factor based" approach

We suggest a new approach, which is numerically stable but requires the following assumptions.

Assumptions

a) The set of roots for a given y $P_y^{-1}(VaR)=\{Z:P(y,Z)=VaR\}$ can be determined and is finite for each y.

b) The cumulative distribution function of the risk factor Z, i.e. F_z^Y for given Y is known and numerically feasible.

c) We know, or have estimated, the overall VaR, to which we want to determine contributions ("allocations").

Remark: It is not necessary to have a full analytical description of the relationship between all risk factors and the change in net asset value. It is however beneficial to have an analytical relationship between the selected main risk factor, given a constellation of remaining risk factors Y, and the change in net asset value, just in order to be able to easily determine $\boldsymbol{P_y^{-1}(VaR)}$. It is also helpful if the distribution of the main risk factor is independent of that of the remaining risk factors Y and this distribution is analytical, this enables to determine $\boldsymbol{F_z^Y}$ easily.

As we see later, the main risk factor Z should also be chosen such that it is the risk factor with the largest impact on P around the VaR.

We can define the Value at risk for a given quantile α and Portfolio $P(X)+hP^{(m)}(X)$, VaR(h) by:

$$Prob\left(P(X)+hP^{(m)}(X)\le VaR_\alpha(h)\right)=\alpha$$

For sufficiently well behaved distributions we have:

$$\begin{aligned}&Prob\left(P(X)+h\cdot P^{(m)}(X)\le VaR_\alpha(h)\right)\\&=\int_{\Omega_Y}\int_{\Omega_Z}1_{P(Y,Z)+h\cdot P^{(m)}(Y,Z)\le VaR_\alpha(h)}dF_Z^Y\,dF_Y\\&=\int_{\Omega_Y}(\textstyle\sum_i S_i\int_{Root(i,Y,h)}^{Root(i+1,Y,h)}dF_Z^Y)\,dF_Y\end{aligned}$$

where $1_{a\le b}$ is 1, if a≤b and 0 otherwise, Root(i,Y,h) is the i-th root of $P(Y,Z)+h\cdot P^{(m)}(Y,Z)-VaR_\alpha(h)$ in Z, for given Y and h and the sum is taken over all such roots. S_i is 1 if $P(Y,Z)+h\cdot P^{(m)}(Y,Z)-VaR_\alpha(h)$ is negative on $[Root(i,Y,h),Root(i+1,Y,h)]$ and 0 otherwise. Further on $Root(0,Y,h):=-\infty$ and the last root is ∞. F_z^Y and F_Y are the distribution functions of Z given Y and Y respectively.

Therefore, for v≠0

$$\begin{aligned}0&=\tfrac{\partial}{\partial h}\int_{\Omega_Y}(\textstyle\sum_i S_i\int_{Root(i,Y,h)}^{Root(i+1,Y,h)}dF_Z^Y)\,dF_Y\,|_{h=v}\\&=\int_{\Omega_Y}(\textstyle\sum_i S_i\tfrac{\partial}{\partial h}\int_{Root(i,Y,h)}^{Root(i+1,Y,h)}dF_Z^Y\,|_{h=v})\,dF_Y\\&=\int_{\Omega_Y}\left(\textstyle\sum_i S_i\,(C_{(i+1)}-C_{(i)})\right)dF_Y\end{aligned}$$

with

$$C_{(i)}):=f_Z^Y\,|_{Z=Root(i,Y,v)}\tfrac{\partial}{\partial h}Root(i,Y,h)\,|_{h=v}$$

Where f_Z^Y is the density associated with F_Z^Y.

Set:

$$\begin{aligned}&R_{(i,h)}:=Root(i,Y,h)\\&V_{(h)}:=VaR_\alpha(h)\\&\mathrm{P}(i,v):=P^{(m)}\left(Y,Root(i,Y,v)\right)\end{aligned}$$

By definition:

$$0=P(Y,R_{(i,h)})+hP_{(i,h)}-\mathrm{V}(h)$$

Thus:

$$\begin{aligned}0&=\tfrac{\partial}{\partial h}(P(Y,R_{(i,h)})+hP_{(i,h)}-V_{(h)}\,|_{h=v}\\&=\tfrac{\partial}{\partial h}R_{(i,h)}\,|_{h=v}\,\tfrac{\partial}{\partial Z}\,P(Y,Z)(R_{(i,v)})\\&\quad+v\tfrac{\partial}{\partial h}P_{(i,h)}\,|_{h=v}+P_{(i,v)}-\tfrac{\partial}{\partial h}V(h)\,|_{h=v}\end{aligned}$$

Therefore

$$\frac{\partial}{\partial h}R_{(i,h)}\Big|_{h=v}=\frac{-P_{(i,v)}-v\cdot\frac{\partial}{\partial h}P_{(i,h)}\big|_{h=v}+\frac{\partial}{\partial h}V_{(h)}\big|_{h=v}}{\frac{\partial}{\partial Z}P(Y,Z)\left(R_{(i,v)}\right)}$$

$$\frac{\partial}{\partial Z}P(Y,Z)\left(R_{(i,v)}\right)\neq 0.$$

If $\frac{\partial}{\partial Z}P(Y,Z)\left(Root(i,Y,v)\right)=0$, then there is no change of sign of $P(Y,Z)$ at $Root(i,Y,v)$ and we can omit i in the summation over the roots. We assume without loss of generality that this is the case.

Combining these results we have:

$$0=\int_{\Omega_Y}\left(\textstyle\sum_i S_i\,(C_{(i)}\frac{A_{(i+1)}}{B_{(i+1,v)}}-C_{(i)}\frac{A_{(i)}}{B_{(i,v)}}\right)dF_Y$$

with

$$A_{(i)}:=-P_{(i,v)}-v\cdot\tfrac{\partial}{\partial h}\left(P_{(i,h)}+VaR_\alpha(h)\right)|_{h=v}$$

$$B_{(i,v)}:=\tfrac{\partial}{\partial Z}P(Y,Z)\left(Root(i,Y,v)\right)$$

$$C_{(i)}:=F_z^Y\,|_{Z=Root(i,Y,v)}$$

$$P_{(i,v)}:=P^{(m)}\left(Y,Root(i,Y,v)\right).$$

Now we consider the case when v->0, and note that the roots are heavily v-dependent, but not the number of roots, for v sufficiently close to 0. We also assume that the roots are continuous on v for fixed Y.

Obviously, if $\frac{\partial}{\partial Z}P(Y,Z)\left(Root(i+1,Y,0)\right)$ is positive then $P(Y,Z)-VaR_\alpha$ is negative on $[Root(i,Y,0),Root(i+1,Y,0)]$ and thus $S_i=1$. If $\frac{\partial}{\partial Z}P(Y,Z)\left(Root(i+1,Y,0)\right)$ is negative, then $S_i=0$. And we know that the signs of $\frac{\partial}{\partial Z}P(Y,Z)(Root(i,Y,0))$ are strictly alternating here. Thus:

$$0=\int_{\Omega_Y}\left(\textstyle\sum_i S_i\,(C_{(i+1)}\frac{A_{(i+1)}}{B_{(i+1,v)}}-C_{(i)}\frac{A_{(i)}}{B_{(i,0)}})\right)dF_Y$$

$$=\int_{\Omega_Y}(\textstyle\sum_i C_{(i)}\frac{A_{(i)}}{|B_{(i,0)}|})\,dF_Y$$

with

$$A_{(i)}:=-P_{(i,0)}+\tfrac{\partial}{\partial h}VaR_\alpha(h)\,|_{h=0}$$

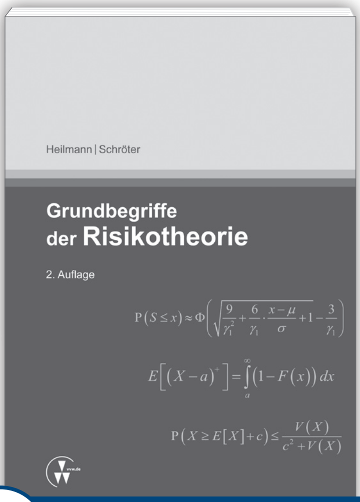
$$B_{(i,v)}:=\tfrac{\partial}{\partial Z}P(Y,Z)\left(Root(i,Y,v)\right)$$

$$C_{(i)}:=F_z^Y\,|_{Z=Root(i,Y,0)}$$

$$P_{(i,v)}:=P^{(m)}\left(Y,Root(i,Y,v)\right).$$

Therefore:

$$\frac{\partial}{\partial h}VaR_\alpha(h)\Big|_{h=0}=\frac{\int_{\Omega_Y}\left(\sum_i\frac{C(i)P_{(i,0)}}{|B_{(i,0)}|}\right)dF_Y}{\int_{\Omega_Y}\left(\sum_i\frac{C(i)}{|B_{(i,0)}|}\right)dF_Y}$$



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We now approximate the integrals, which in fact are expected values, by stochastic simulation, as usual:

$$C_m = \frac{\partial}{\partial h} VaR_\alpha(h)\Big|_{h=0} \approx \frac{\Sigma_j\left(\Sigma_i \; D_{(i,j)}\right)}{\Sigma_j\left(\Sigma_i \; E_{(i,j)}\right)}$$

$$\text{with } D_{(i,j)} := \frac{f_Z^{y^j}\Big|_{Z=Root(i,y^j,0)} P^{(m)}\left(y^j, R_{(i,j,0)}\right)}{\left|\frac{\partial}{\partial Z} P(Y,Z)\left(R_{(i,j,0)}\right)\right|}$$

$$E_{(i,j)} := \frac{f_Z^{y^j}\Big|_{Z=R_{(i,j,0)}}}{\left|\frac{\partial}{\partial Z} P(Y,Z)\left(R_{(i,j,0)}\right)\right|}$$

$$R_{(i,j,h)} := Root(i, y^j, h)$$

As we can see the integrals are now evaluated scenario-wise and the dimension of the integration has been reduced by 1 at the cost of determining roots and evaluating the conditional density.

The quality of the estimator is the better the lower the volatility of

$$\Sigma_i \; \frac{f_Z^{y^j}\Big|_{Z=Root(i,y^j,0)} P^{(m)}\left(y^j, Root(i,y^j,0)\right)}{\left|\frac{\partial}{\partial Z} P(Y,Z)\left(Root(i,y^j,0)\right)\right|} \text{ and}$$

$$\Sigma_j\left(\Sigma_i \; \frac{f_Z^{y^j}\Big|_{Z=Root(i,y^j,0)}}{\left|\frac{\partial}{\partial Z} P(Y,Z)\left(Root(i,y^j,0)\right)\right|}\right) \text{ is.}$$

A necessary condition for a low volatility is that these sums are not taken over empty index-sets often, i.e. that for many chosen simulations Y_i there are roots of $P(Y,Z) - VaR_\alpha$.

This is the case if we choose the risk factor X_n to be the risk factor with the largest variability and impact, i.e. potential to create solutions of $P(Y,Z) - VaR_\alpha$.

So the process is as follows:

1) Draw a sufficient number s of simulations for the risk factors $y^i = (x_1^i, x_2^i, \ldots, x_{n-1}^i)$, i=1,…,s.

2) Determine $P^{-1}(y^i, VaR)$ for each of these. Here it helps if P is a polynomial of sufficiently low degree.

And we remember the reader, that VaR must be known and is regarded as a constant here.

3) Determine

$$C_m := \frac{\Sigma_j\left(\Sigma_i \; K_{(i,j)} P^{(m)}\left(y^j, Root(i,y^j,0)\right)\right)}{\Sigma_j\left(\Sigma_i \; K_{(i,j)}\right)}.$$

Obviously:

$$\sum C_m = \sum \frac{\Sigma_j\left(\Sigma_i \; K_{(i,j)} P^{(m)}\left(y^j, Root(i,y^j,0)\right)\right)}{\Sigma_j\left(\Sigma_i \; K_{(i,j)}\right)}$$

$$= \frac{\Sigma_j\left(\Sigma_i \; K_{(i,j)} \underbrace{\sum P^{(m)}\left(y^j, Root(i,y^j,0)\right)}_{VaR}\right)}{\Sigma_j\left(\Sigma_i \; K_{(i,j)}\right)} = VaR$$

$$\text{with } K_{(i,j)} := \frac{f_Z^{y^j}\Big|_{Z=Root(i,y^j,0)}}{\left|\frac{\partial}{\partial Z} P(Y,Z)\left(Root(i,y^j,0)\right)\right|}.$$

It might now seem difficult to determine the conditional probability density $f_Z^{y^j}\big|_{Z=Root(i,y^j,0)}$ for (y^i, z) with the y^is given, since the multivariate distribution of the risk factors might be complex, or even based on causal structures, not allowing to determine this probability analytically.

But in all cases known to the author the generation of the simulations $x^1,\ldots,x^s$ are generated by creating k-dimensional independent standard normal simulations $r^1,\ldots,r^s$ and deriving the risk factor simulations from this using a well-defined function RF, such that: $RF(r^i) = x^i$, i = 1,…,s.

The only thing we need to do in this case is change our point of view: We just apply our approach using the functions $P^{(1)}(RF(R))$, $P^{(2)}(RF(R)), \ldots, P^{(m)}(RF(R))$ and consider the R as risk factors.

Certainly we know how to calculate the conditional probability density $f_Z^{y^j}\big|_{Z=Root(i,y^j,0)}$ with given y^i for a multivariate distribution of independent normal variables.

If the roots of the functions $P^{(i)}(RF(R))$ turn out to be difficult to determine we just can use a LSMC approach to fit polynomials for the net value to the standard normal simulations $r^1,\ldots,r^s$.

Case study

For the case study we choose the following setting:

1) Three risk factors X ,Y and Z with multivariate normal distribution expressed by a covariance matrix C and expected value 0.

2) 10'000 simulations and 100 experiments, i.e. different esti-

mations of the Euler-allocation using new sampled random numbers.

3) Three polynomials in these 3 risk factors representing three entities, P1, P2 and P3. The parent entity is the sum of these three polynomials. This is of course an example, which is a little bit artificial, because if we have such a nice setting, we could as well generate enough simulations forcing the naïve approach to converge. In such a setting the application of the new approach only makes sense if the gain in convergence speed is not outweighed by the loss in calculation speed caused by the necessity to determine the roots of the polynomials considered. This for example is the case if the polynomials have sufficiently low degree.

4) A = 99.5 % quantile for determining the VaR

5) Risk factor with analytic distribution used for the new approach: z

6) We compare the results derived using the above described approach with the naïve approach, which is using the average of the respective subportfolio associated with 2N simulations of the parent portfolio around the VaR as approximation for the Euler-allocation.

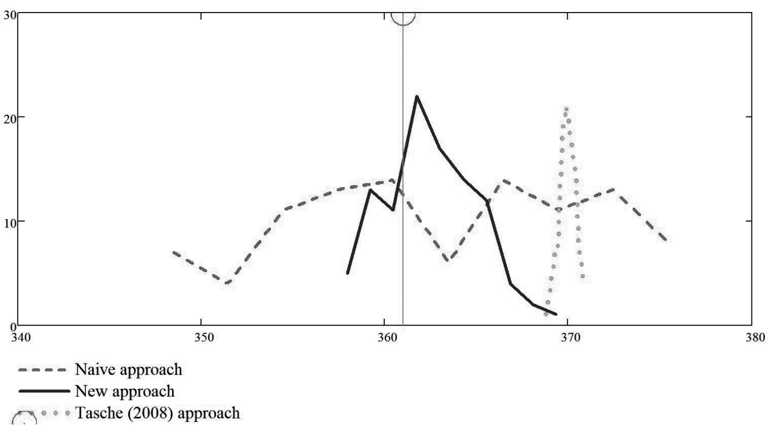
7) We observe that there is another interesting approach to determine the Euler-allocation avoiding the deficiencies of the standard approach. This is the approach described in Tasche (2008), formula (3.9). We also give the results of this approach.

Results:

Harmless case:

$$C=\begin{pmatrix}1.5 & 0.5 & -0.5\\ 0.5 & 1.5 & 0.5\\ -0.5 & 0.5 & 1.5\end{pmatrix}$$

Figure 1:



$P1(x,y,z) := 300 + 20y + 10z + 30x + yx^2$
$P2(x,y,z) := 200 - 10y - 20z - 20x$
$P3(x,y,z) := 300 + 15y + 15z + 5x + 0.1yx^2$

$N = 3$

The graph shows the results sorted into 10 equally-spaced buckets and we can clearly see that the new approach is superior. The x-axis is the estimation of the Euler-allocation in the experiments and the y-axis shows how many experiments yielded results corresponding to the value on (or near) the value given by the x-axis.

The Tasche 2008-approach suffers from the fact that it introduces a bias, as the "steepness"-factor in Tassche's linear formula, $\frac{Cov(P(X),P^{(m)}(X))}{Var(P(X))}$, is determined over the whole range of the scenarios, not focusing on the area around the VaR. Restricting to this area however produces too large an estimation error for the factor $\frac{Cov(P(X),P^{(m)}(X))}{Var(P(X))}$. However I expect this approach to work much better, if this approach is extended to a non-linear approach, as indicated by Tasche.

The volatility of the Euler-allocations of the three portfolios is:

Volatility(Naïve approach, P1)=5.1
Volatility(New approach, P1)=3.326
Volatility(Naïve approach, P2)=7.24
Volatility(New approach, P2)=1.748
Volatility(Naïve approach, P3)=7.945
Volatility(New approach, P3)=2.558

Complex case:

Same parameters as above, except:

$P1(x,y,z) := 300 + 20y + 10z + 30x$
$P2(x,y,z) := 200 - 10y - 15z^2 - 10x$
$P3(x,y,z) := 300 + 15y + 15z^2 - 5x$

Results: Figure 2

Volatility(Naïve approach, P1)=3.341
Volatility(New approach, P1)=2.465
Volatility(Naïve approach, P2)=12.65
Volatility(New approach, P2)=2.579
Volatility(Naïve approach, P3)=13.926
Volatility(New approach, P3)=2.893

Case with wide dispersion of the risk factor z:

Parameters as in the harmless case, except:

$$C=\begin{pmatrix}1.5 & 0.5 & -0.5\\ 0.5 & 1.5 & 0.5\\ -0.5 & 0.5 & 5\end{pmatrix}$$

Results: Figure 3

Figure 2:

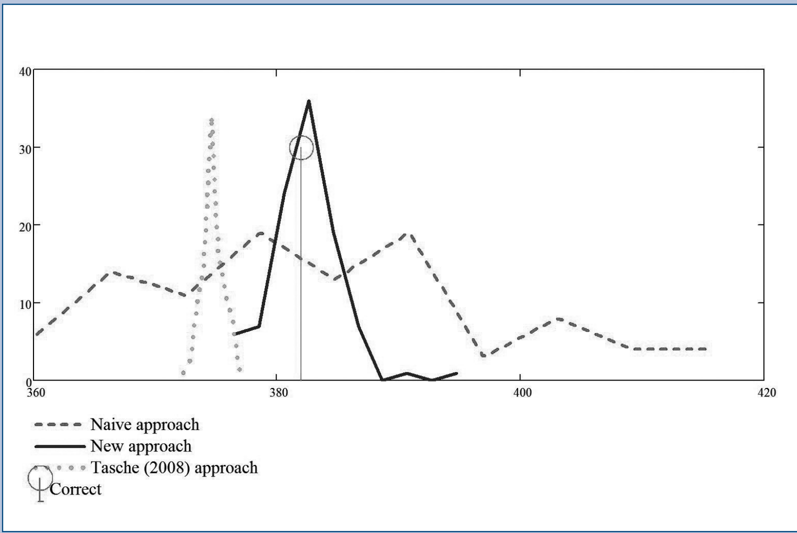
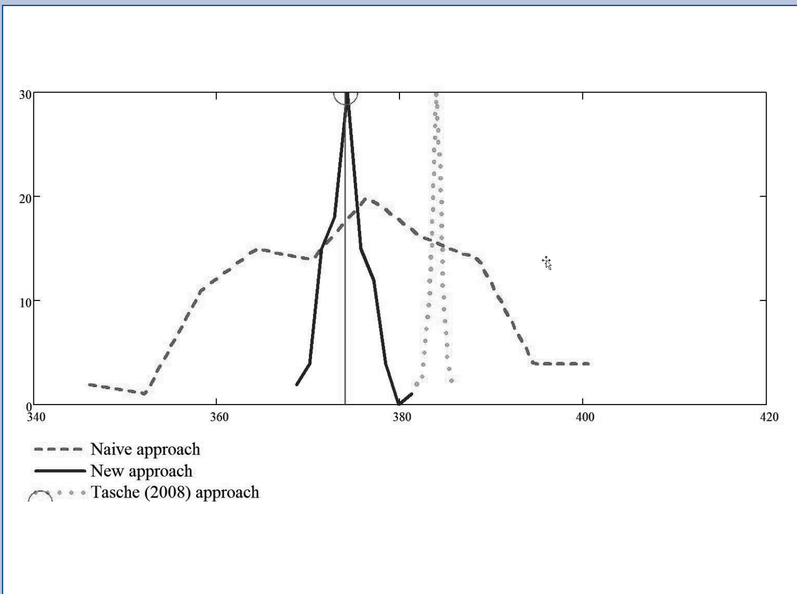


Figure 3:



Volatility(Naïve approach, P1)=5.942
Volatility(New approach, P1)=3.006
Volatility(Naïve approach, P2)=15.306
Volatility(New approach, P2)=2.443
Volatility(Naïve approach, P3)=12.85
Volatility(New approach, P3)=2.364

Volatility(Naive_approach_P1)=8.445
Volatility(New_approach_P1)=2.684
Volatility(Naive_approach_P2)=23.278
Volatility(New_approach_P2)=3.509
Volatility(Naive_approach_P3)=17.666
Volatility(New_approach_P3)=2.816

Apart from the massively improved convergence of the approach the robustness of the approach is superior as it does not produce biased results. Indeed if the VaR changes not linearly for simulations near the VaR then the naïve approach obviously converges to values which are inconsistent with the true Euler allocations as it is a symmetric approach. The naïve approach is a biased approach. While in most cases this might be a negligible effect it is hard to assess in reality whether this effect can be neglected. In contrast, the new approach is unbiased and robust.

Literature

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Tigran Kalberer leitet die Aktivitäten von Milliman, einer der größten aktuariellen Beratungsunternehmen weltweit, in den nordischen Ländern, Deutschland, der Schweiz und Österreich. Er spezialisiert sich auf Risikomanagement, Industrialisierung aktuarieller Prozesse, strategische Fragestellungen für Versicherungsunternehmen inklusive M&A und Interne Modelle. Er hat an der Universität zu Köln Mathematik studiert. Er ist Mitglied des DAV und des SAV.