Construction of an acceleration algorithm of the Nested Simulations method for the calculation of the Solvency II economic capital

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ABSTRACT

The Nested Simulations (NS) method is at the present moment, for the life insurance portfolios, one of the economic capital calculation methods that is most in accordance with the Solvency II guidelines. However, this approach leads to a substantial calculation time, up to a point which could jeopardize its applicability by some insurance companies.

The algorithm that we present in this article allows to significantly reduce the number of simulations applied in the framework of an NS projection. The calculation automaton that we have developed, named the "NS-accelerator", is based on the idea of "locating", throughout the usage of risk factors, the situations that are the most adverse in terms of solvency.

The projection time reduction that is obtained from the application of the NS-accelerator makes hereafter the Nested Simulations approach conceivable no matter what the size and the structure of the company considered is.

KEYWORDS: economic capital, Solvency II, Nested Simulations, economic balance sheet, risk factors, simulation number reduction

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1. Introduction

The economic capital definition in the Solvency II framework relies on the notion of equity[1] distribution over one year. However for most of the insurance portfolios it is difficult to obtain this distribution given the complex dependant relations between the asset and the liabilities that result from the profit sharing mechanisms and from the policyholders' dynamic lapses behaviors. The most accurate economic capital calculation method becomes then to perform "market consistent" simulations (secondary simulations) inside of "real world" simulations of the first year (primary simulations) with the objective of disposing of a the company's balance sheet distribution at t=1. This procedure (used for example by Gordy and Juneja (2008)) is known as Nested Simulations (NS). Nevertheless, this approach requires a great number of simulations and leads to a substantial calculation time. Numerous techniques permit to improve the speed of the simulations convergence in a Monte Carlo application. These methods, in particular those based on a system of particles in interaction, are described in the books by Del Moral (2004) and Glasserman (2004) for the financial engineering applications.

In this article, we present a calculation algorithm that allows reducing in a very significant manner the number of simulations in an NS projection. While before the standard NS methodology was hard to implement in some companies (in particular in function of the number of portfolios), the algorithm that we have developed makes hereafter the NS approach conceivable no matter what the size or the structure of the company considered is.

This method's principle, that we will call the "NS-accelerator", consists of "locating" the configurations of scenarios risk factors that are the most adverse in terms of solvency.
The NS-accelerator is based on three key steps:

- the extraction of the elementary risk factors that bear the risk intensity of the primary simulations,
- construction of the execution regions,
- an iterative procedure on the execution region's thresholds in order to give priority to the calculations related to the situations that are the most adverse in terms of solvency.

In the first part, we discuss the problems related to the economic capital calculation in a Solvency II perspective. We will elaborate thereafter on the NS methodology followed by a detail description of the acceleration algorithm that we have developed. In the last section we will present the results obtained with the NS-accelerator on a savings portfolio.

[1] We will use the term "equity" to refer to the economic value of the company's Net Worth. To refer to the part of the company's assets composed of shares of stock we will use the term "stock".

2. The economic capital in the Solvency II framework

2.1. Definition

In the Solvency II framework, the economic capital corresponds to the amount of equity of which the company must dispose to confront an economic ruin in a 1 year time horizon at a confidence level of 99,5%.
This definition of economic capital is based on three notions:

- The **economic ruin** corresponds to a situation where the company's asset market value is less than the liability fair value. In other terms, it is a situation in which the equity of the company (calculated as the difference between the total assets' market value and the total fair value of the liabilities) is negative.
- **The one year time horizon** imposes being able to distribute equity in a year period. Observe that in fact the equity of the company in the initial date constitutes a deterministic amount while a year later it becomes a random variable. The random variable value depends on different sources of randomness (financial, demographic ...) that are attained during the year.
- **The 99.5% threshold** represents the required solvency level. The probability of arriving to an "economic ruin" in this case is less than 0.5%.

2.2. Theoretical formalization

We will introduce in this section the set of notations and concepts that we will use in the article.

2.2.1. Economic balance sheet

The capital definition in the solvency II environment is based on an economic vision of the company's balance sheet. At a date *t*, it is possible to construct an economic balance sheet as follows:

Economic Balance Sheet at t

A_t	E_t L_t

Under the notation:

- A_t : asset market value at time *t,*
- L_t *:* fair value of the liabilities at time *t,*
- E_t : equity at time *t.*

As the balance sheet is in equilibrium, we have the relation: $E_t = A_t - L_t$.
Note by $(F_t)_{t\geq 0}$ the filtration that characterizes the available financial information at each date.

The value of each balance sheet part corresponds to the expected value of the discounted future cash-flows under a risk-neutral probability Q.

Let:

- δ_u be the discount factor expressed in terms of a risk free instantaneous interest rate r_u : $\delta_u = e^{-\int_0^u r_h dh}$,
- P_t be the liability cash-flows (claims, commissions, fees, ...) in period t,
- R_t be the company's profit in period t.

Under this notation, the equity E_0 and the liability fair value L_0 at the initial date are calculated in the following manner:

$$L_0 = E_Q\left[\sum_{u\geq 1} \delta_u . P_u \,|F_0\right],$$

and,

$$E_0 = E_Q\left[\sum_{u\geq 1} \delta_u . R_u \,|F_0\right].$$

2.2.2. The "real world" and risk free probabilities

In the Solvency II framework, the insurer must dispose of enough equity at the initial date in order to satisfy the following condition: $P(E_1 < 0) \leq 0{,}5\%$.

The ruin probability is based on the event $\{E_1 < 0\}$. This assumes that the first period calculations are necessarily done in a "real world" universe. Indeed to correctly assess this probability, the random events occurring during the first period must evolve in the same fashion as the movements observed in reality.

Even though the "real world" probability is used during the first year, it is consistent, in this logic, to use a "market consistent" approach to calculate the balance sheet parts henceforth, that is, by using the expected value under the risk-free probability of the discounted future flows (supposing that one exists, and that it is unique, or that one can be chosen using a coherent method). This brings us to the following relations:

- $E_1 = R_1 + E_Q\left[\sum_{u\geq 2} \frac{\delta_u}{\delta_1} R_u \,|F_1^{RW}\right]$,
- $L_1 = E_Q\left[\sum_{u\geq 2} \frac{\delta_u}{\delta_1} P_u \,|F_1^{RW}\right]$,

where F_1^{RW} represents the « real world » information at the end of the first year.

2.2.3. The economic capital calculation

The economic capital, as defined by Solvency II, is the minimal level of equity at the initial date that is needed in order to have: $P(E_1 < 0) \leq 0{,}5\%$.

To estimate the economic capital the following relation is used: $C = E_0 - P(0{,}1).\, q_{0{,}5\%}(E_1)$, where P(0,1) is the price at 0 of a zero-coupon bond that matures in a year.

The quantity $-P(0,1). q_{0,5\%}(E_1)$ then becomes an (algebraic) surplus that is necessary to be added to the initial equity in order to satisfy the condition $P(E_1 < 0) = 0{,}5\%$.

Suppose that the amount $S = -P(0,1). q_{0,5\%}(E_1)$ is invested (or short sold) in cash and evolves during the first year at a risk free return.
Note E_1^{ajust}, A_1^{ajust}, L_1^{ajust} the economic variables after incorporating S. It is easy to show that after including the capital, the solvency condition at a 99,5% threshold is verified:

$$
\begin{aligned}
&P\left(E_1^{ajust} < 0\right) \\
&= P\left(A_1^{ajust} - L_1^{ajust} < 0\right) \\
&\approx P\left(A_1 + \frac{S}{P(0,1)} - L_1 < 0\right) \qquad (*) \\
&= P\left(E_1 + \frac{S}{P(0,1)} < 0\right) \\
&= P\left(E_1 - q_{0,5\%}(E_1) < 0\right) \\
&= P\left(E_1 < q_{0,5\%}(E_1)\right) \\
&= 0{,}5\%.
\end{aligned}
$$

The (*) approximation is based on the assumption that the participation, in terms of cash, of the capital S in the whole value of the first year results and in the conditional subsequent results is negligible. Observe that in the case where liabilities are divided in segregated funds, the equality is satisfied and (*) is not an approximation anymore.

3. The economical capital calculation with the NS approach

3.1. The stochastic calculation in the “life” ALM models

The projection tools in the life insurance portfolios are in the real practice fed with stochastic scenario tables (economic scenarios, mortality scenarios...). These tables are previously generated by applications constructed for such a purpose. An economic scenarios table contains for example, for each simulation, the evolutions of the interest rate curve, the equity index, the inflation index, etc. It must be emphasized that in a “life” ALM model, a scenario corresponds to a deterministic projection of the whole set of actuarial elements (balance sheet parts and income statement) calculated based on a particular scenario. Therefore, in what follows of the present article, the term “simulations” will be used without distinction to refer to the scenario tables and to the calculations of the subsequent actuarial elements.

3.2. The NS approach description

The distribution of the equity value of the company at t=1 from an internal model perspective, is based on a methodology called Nested Simulations (NS).

The Solvency II framework requires to consider simultaneously in the first period a "real-world" evolution of the random variables modeled (risk approach) and a "market consistent" valuation (economic approach) of the balance sheet parts at the end of the first year.

The asset market value of the company at the end of the first period is often very quick to calculate *with the models used by the companies*. In fact the pricing of the different assets is often supported on closed formulas of Black or Black & Scholes type for the derivative products, and the stock and bond values are obtained directly from the stock index and interest rate curve at the end of the first year, which are contained in the economic scenarios table. Since some liabilities have a long duration, the Black or Black & Scholes formulas must be used with caution or must be adapted. However, in this article we will focus only on the time reduction problems. We will therefore consider the most frequently used models for this type of risks in order to illustrate our results, while nonetheless mentioning the precautions that need to be kept in mind to use our method on models which are closer to the quantification of real risks.

In general, the liability fair value and the equity value at the end of the first period are tougher to obtain (*again in the models used by the companies*). In fact, the asset/liability dependency relations that arise from the profit sharing mechanisms and from the policyholders' lapses behaviors interfere with the implementation of "closed-formulas". Therefore it is necessary to resort to Monte Carlo type methods in order to estimate the liability fair value and therefore the equity value at the end of the first year.

In order to do so, by the means of an internal model, "real world" simulations are carried out over the first year (called primary simulations). Henceforth, starting from each of the previous simulations, a new set of simulations is carried out (secondary simulations) in order to establish the company's equity. The secondary simulations' purpose is to price the balance sheet elements, they must therefore be "market-consistent"; very often they are "risk-neutral" simulations.

This implementation can be illustrated with the following diagram:

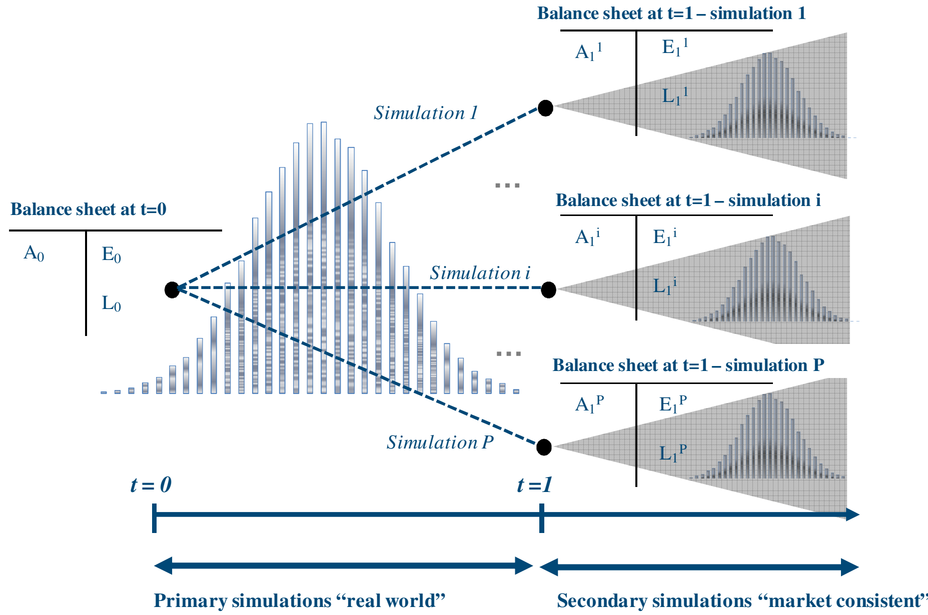

Figure 1 : Equity distribution calculation in the NS methodology

Each of the set of secondary simulation must be conditioned by the corresponding primary simulation, i.e. by the "real world" information from the first period.

For the NS approach calculations use the following notations:

- $R_u^{p,s}$ profit at the date $u > 1$ for the primary simulation $p \in \{1, \dots, P\}$ and the secondary simulation $s \in \{1, \dots, S\}$,
- R_1^p the profit from the first period of the primary simulation p,
- $P_u^{p,s}$ the liability cash-flows at date $u > 1$ for the primary simulation p and the secondary s,
- $\delta_u^{p,s}$ the discount factor at the date $u > 1$ for the primary simulation p and secondary s,
- δ_1^p the discount factor for the first period for the primary simulation p,
- F_1^p the first year information contained in the primary simulation p,
- E_1^p the equity at the end of the first period for the primary simulation p,
- L_1^p the liability fair value at the end of the first period for the primary simulation p.

The balance sheet parts at $t = 1$, for the primary simulation p, are therefore calculated in the following manner :

$$E_1^p = R_1^p + E\left[\sum_{u\geq 2} \frac{\delta_u}{\delta_1} R_u \,\middle|\, F_1^p\right] \approx R_1^p + \frac{1}{S}\sum_{s=1}^{S}\sum_{u\geq 2}\frac{\delta_u^{p,s}}{\delta_1^p} R_u^{p,s},$$

and,

$$L_1^p = E\left[\sum_{u\geq 2} \frac{\delta_u}{\delta_1} P_u \,\middle|\, F_1^p\right] \approx \frac{1}{S}\sum_{s=1}^{S}\sum_{u\geq 2}\frac{\delta_u^{p,s}}{\delta_1^p} P_u^{p,s}.$$

Remember that the secondary simulations have as main purpose the estimation of the conditional expected value of E_1^p and L_1^p while the primary simulations are used to obtain the empirical distribution of the random variable E_1 for the calculation of the 0,5% percentile. Consequently, the number of secondary simulations is generally much lower (usually about 1000 simulations) than the number of primary simulations (at least 5000 in practice). Lee's doctoral dissertation (1998) contains interesting results on the estimation of conditional expectations.

The $P \times S$ complexity leads to very significant calculation times. It is therefore necessary to develop an acceleration algorithm that allows to considerably reduce the number of simulations required. To reduce the number of secondary simulations we use the classical variance reduction techniques (antithetic variables, simultaneous use of Sobol sequences and Brownian bridges...); this implementation improves the "empirical mean" estimators of the E_1^p and L_1^p convergence. The NS accelerator that we describe in the following section is a calculus automaton that by himself allows to reduce the number of simulation in the first period.

4. The NS-accelerator

In this section we will elaborate on the algorithm to accelerate the NS projection.

4.1. Introduction

4.1.1. Core issue

The goal of a NS calculation is to obtain the 0,5% percentile of the equity E_1 distribution at $t = 1$. Therefore, only the tail of this distribution is useful to establish the economic capital.

Suppose, for example, that we perform an NS projection based on 5000 primary simulations. We then sort by increasing order the 5000 realizations of the E_1 variable in order to dispose of the order statistics $\left(E_1^{(p)}\right)_{p=1,\dots,P}$.

To determine the $q_{0,5\%}(E_1)$ percentile, it is a usual method to employ the estimator $E_1^{(0,5\% \times 5000)} = E_1^{(25)}$. In other terms, we chose the 25th "worst value" of the $\left(E_1^p\right)_{p=1,\dots,P}$ sample as an estimator of $q_{0,5\%}(E_1)$.

In conclusion, in order to establish the economic capital at a 0,5% level from the P primary simulations, it is then not necessary to dispose of the whole empirical distribution of the variable E_1 but only of the $[0,5\% \times P]$ worst values of the $\left(E_1^p\right)_{p=1,\dots,P}$ sample.

4.1.2. Characterization of the worst case scenarios

In order to speed up the calculations and to consider the situations that are the most adverse in terms of solvency, it becomes necessary to measure the risk intensity related to each simulation so that the projections can be organized in order of priority.

To illustrate this notion, we consider in this section (and in what remains of the article) the case of a standard savings portfolio[2] which has a risk exposure to the interest rate rise (zero minimum guaranteed rate and with a dynamic lapse behavior) in which we have carried out a thorough NS projection (5000 primary simulations) in order to establish the economic capital due to market risk[3].

In this portfolio, a simulation in which the stock index has a high performance while the interest rates slightly decrease cannot lead to one of the 25 worst equity values. Therefore carrying out this simulation does not provide any information in order to measure the percentile but instead it wastes calculation time.

However this approach has to be able to resume each of the underlying risks (stock, interest rate, mortality, ...) in relatively simple factors that allow to translate the risk intensity carried out by each simulation[4]. We will see with more detail in the section dedicated to the construction of risk factors how each simulation can be synthesized by an n-tuple of factors that characterize the intensity of each risk.

The following graph represents, in the context of our example, the stock and bond risk factors (refer to the sections to come for a definition of these risk factors) that allow to synthesize each of the 5000 primary simulations by factor couples.

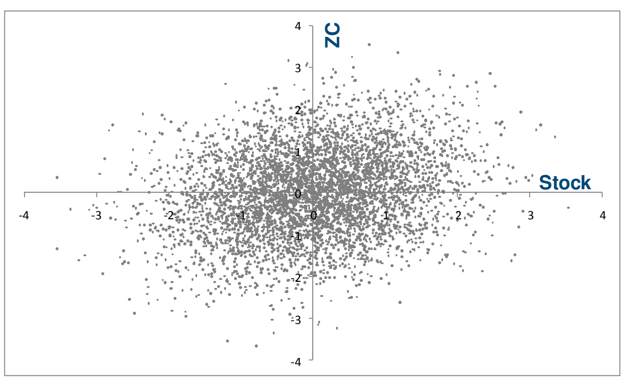

Figure 2 : Risk factor couples, stock (on the abscissa) × zero-coupon bonds (on the ordinate), that represent the 5000 primary simulations.

The realization of a thorough NS projection has allowed us to relate to each point a level of equity at the end of the first period.

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[2] This example will also be used in the section dedicated to the implementation of the NS accelerator.

[3] We have considered a market risk only composed of stock and bond risks.

[4] Observe that it is enough to "resume" the stock variability by its performance, while for the interest rate risk the analysis should be done by maturities.

If we track the 50 situations that are the most adverse in terms of solvency we observe that these points appear in the outer area as of the point set, as it can be observed in the following graph:

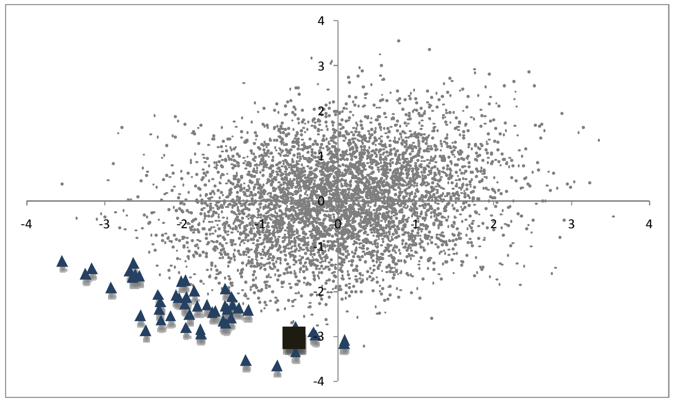


Figure 3 : the triangles correspond to the 50 couples of risk factors that are the most adverse - the square corresponds to the scenario related to the 25th worst value of E_1 -

Once we define in a mathematical manner the notion of outer area, that is by introducing a notion of mathematic norm (distance to the origin), it becomes possible to carry out the analysis and the calculations on the simulations that are the most adverse.

In this example and in this model, the most adverse scenarios are indeed located in the outer region of the point set because the risk factors have been identified and used. However, in other cases, for example if a particularly unfavorable change of trend occurs at the very end of the first year, the most adverse scenarios could be located elsewhere than the outer region. Let us consider the global risk associated to a death insurance portfolio. A natural choice of factors could correspond to a set of annual death rate (for different ages for example), to which we would add two market risk factors (equity and bonds). In this case the most adverse scenarios would most likely be those associated to the occurrence of a pandemic. Yet if a pandemic occurred at the very end of the civil year, and did not have enough time to significantly degrade the death rate of this year, this could lead to one the scenarios which are to be selected, being among the most adverse, while it would be represented by a point located in the inner region of the point set (if equity and bonds do not have the time to fall either). To avoid such phenomena preventing the NS-accelarator from selecting the most adverse scenarios, it is necessary to pick the right risk factors: in this precise case, it is inevitable to add an “intensity of the pandemic in progress” factor (with a very high level in the case of an absence of a pandemic). Once this factor is added, any ongoing pandemic will correspond to a point with a very negative coordinate, therefore located in the outer region. In this precise case, we could also use all kinds of geometric quantiles (see Barnett(1976)) with non-elliptic level hyper-surfaces(allowing to classify scenarios with a pandemic component as most adverse, even though the other factors are not alarming). The difference with traditional methods for the estimation of geometric quantiles (see in particular Barnett(1974) and Chaouch(2008)), is that we do not know very well the function which is to be applied to the risk factors, and that we obtain it by simulation, which can introduce further numerical problems.

4.1.3. The key ideas

The NS-accelerator presented in this article is nothing else than a calculation automaton that decides which primary simulations should be performed in function of the their risk level. Furthermore, the automaton links each scenario of the first period with a norm that represents the level of adversity and then performs the simulations in order, starting from the worst, up to the moment it obtains the N "worst[5] values" of equity in an stable manner.

The method follows several steps:

- *Step 1*: construction of the elementary risk factors (stock, interest rate, mortality...) that bear the risk intensity of each primary simulation and the establishment of the associated norm.
- *Step 2:* definition of an execution region associated to the required threshold: when a risk factor belongs to the execution region, the primary simulations are carried out.
- *Step 3:* iteration over the region threshold in order to incorporate at each step a number of additional scenarios to be carried out.
- The algorithm stops when the N "worst values" of equity are stable.

The following diagram illustrates the steps just described:

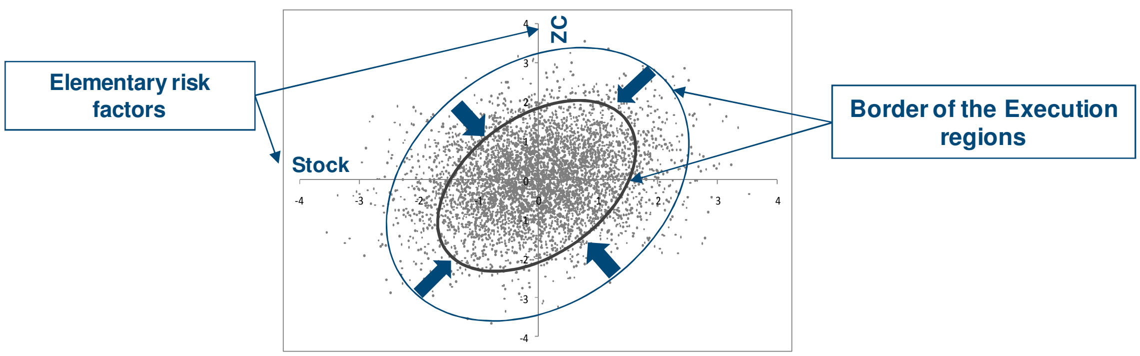


Figure 4 : calculations are run iteratively on successive execution regions

4.2. The elementary risk factors

We will elaborate on this section the different issues related to the risk factor identification and we will describe the construction of the stock and zero-coupon bond risk factors used in this study.

4.2.1. Core Issues

Remember that the risk factors synthesize the risk intensity of each primary simulation. For example, suppose that the stock index is simulated following a geometric Brownian motion, we can use as risk factor the Brownian diffusion movement increment over a period. Large negative values of this

[5] with $N = [0,5\% \times P]$.

increment imply that the stock index has significantly decreased (adverse situation in terms of solvency).

Thus, the risk factor choice depends in the manner the risk has been modeled. Nevertheless, this information is not always known by the company. In fact, many of them resort to external scenarios generators supplied by specialized provider firms that communicate almost none of the structure of their models.
At this point, it becomes necessary to distinguish two types of simulations:

- Case 1: the company knows exactly the models of the underlying risks and simulates the trajectories.
- Case 2: the company does not know with precision the underlying models (structure and parameters) and only disposes of the scenario tables (economic, mortality...).

In the first case, it is enough to export the set of simulated hazards at the moment of primary trajectories generation. We dispose, among others, of the realizations of the Brownian movements of the diffusions (interest rates, stock...); the factors are known a priori.

On the contrary, for the second case, it is necessary to suppose an underlying model for each risk, deduce the parameters from it and thereafter extract the associated risk factors. We will call this approach the “a posteriori inference method”.
In the example presented in the “implementation of the NS-accelerator” section, we have followed an a posteriori approach based on a “real-world” first year table used for the NS projection.

Convention: for a standard analysis, the risk factors inferred will be standardized, that is centered and divided by the standard deviation.

4.2.2. Risk factors vector distribution

We will suppose from here on the following fundamental assumption:
H0 : the risk factor vector is a Gaussian vector with standard normal marginal distributions

This hypothesis supposes at the same time that:

- *H1* : Risk factors follow normal distributions
- *H2* : The vector copula is a normal copula

Remarks:

- Throughout the application of different transformations over the marginal distribution (using the cumulative distribution functions), we can always come to the situation where the distribution of the factors is a **normal standard distribution**.
- The strong assumption is to suppose that the vector copula is normal.

Those hypotheses are obviously questionable, and it is clear that a finer modeling of those complex phenomena would be necessary. However most of insurance companies models use Gaussian random variables. For example, the diffusions of financial models are supported on correlated

Brownian motions, a mortality model of the Lee-Carter type is constituted of a temporal component of the ARIMA type...
The H0 assumption then seems in accordance with the models used in the majority of the present internal models, and the purpose of this work is to improve the feasibility of existing risk calculations as they are performed currently, and not to build a more efficient internal model.

4.2.3. A posteriori factors extraction example

As an example, we will present in this section the treatment of the two risks modeled in our study, that is, stock and interest rate.

Model
In what remains of this section we will use the following notation:
- S_t stock index at time *t,*
- μ_a historic stock index return,
- σ^a historic stock index volatility,
- $P(t,T)$ price at t of an unitary zero-coupon bond with maturity $T > t$,
- μ_T^P historic return for a zero-coupon with maturity *T,*
- σ_T^P historic volatility for a zero-coupon with maturity *T.*

We have assumed that the "real world" evolution of the stock index and the zero-coupon bond prices on the first year can be written as:

$$S_1 = S_0 e^{\mu_a - \frac{1}{2}{\sigma^a}^2 + \sigma^a . Z^a} , \qquad (1)$$

and,

$$P(1,T) = P(0,T) e^{\mu_T^P + \sigma_T^P . Z^{ZC}}, \quad (2)$$

with $Z^a$and Z^{ZC} two standard normal distributions with Person's correlation coefficient $\rho = < Z^a, Z^{ZC} >$.

Observe that the drift and volatility parameters that are used in the relations are historic parameters and not "market consistent" since the economic projections of the first period are "real world".

The relation (1) corresponds to a geometric Brownian motion model of the stock index with constant drift (one year risk free rate plus a risk premium). Furthermore, we have shown that in a HJM (Heath-Jarrow-Morton) model type with linear volatility, the zero-coupon bond price can be expressed following relation (2) [6].

Parameter estimation
Let :
- S_1^p be the stock index at date 1 in a primary simulation p,
- $P^p(1,T)$ be the price at 1 of a unitary zero-coupon bond with maturity T in the simulation *p.*

[6] Refer to the annex.

The parameters were then estimated from the economic scenarios of the first period using the following estimators.

$$\hat{\sigma}^a = \sqrt{\frac{1}{P-1}\sum_{p=1}^{P}\left(ln\left(S_1^p/S_0\right) - \frac{1}{P}\sum_{p=1}^{P} ln\left(S_1^p/S_0\right)\right)^2},$$

$$\hat{\mu}^a = \frac{1}{P}\sum_{p=1}^{P} ln\left(S_1^p/S_0\right) + \frac{1}{2}\hat{\sigma}^{a2},$$

$$\hat{\sigma}_T^P = \sqrt{\frac{1}{P-1}\sum_{p=1}^{P}\left(ln(P^p(1,T)/P(0,T)) - \frac{1}{P}\sum_{p=1}^{P} ln(P^p(1,T)/P(0,T))\right)^2},$$

$$\hat{\mu}_T^P = \frac{1}{P}\sum_{p=1}^{P} ln(P^p(1,T)/P(0,T)).$$

Remark: the values seen at $t = 0$ (i.e. $S_0, P(0,T)$) are deterministic and do not depend on the primary simulation considered.

Risk factor extraction: stock and zero-coupon bonds

For each primary simulation p, we have constructed the couple $(\varepsilon_{Stock}^p, \varepsilon_{ZCB}^p)$ using the estimators previously presented:

$$\varepsilon_{Stock}^p = \frac{ln\left(\frac{S_1^p}{S_0}\right) - \left(\hat{\mu}^a - \frac{1}{2}\hat{\sigma}^{a2}\right)}{\hat{\sigma}^a},$$

(*)

$$\varepsilon_{ZCB}^p = \frac{1}{T-1}\sum_{t=2}^{T}\left[\frac{ln\left(\frac{P^s(1,t)}{P(0,t)}\right) - \hat{\mu}_T^P}{\hat{\sigma}_T^P}\right],$$

$$\hat{\rho} = \frac{1}{P}\sum_{p=1}^{P} \varepsilon_{Stock}^p . \varepsilon_{ZCB}^p .$$

In conclusion, after the extraction process, each primary simulation is characterized by the couple $(\varepsilon_{Stock}^p, \varepsilon_{ZCB}^p)$ that synthesizes the risk intensity of the stock and zero-coupon bond factors.

Remark: a high positive value of ε_{ZCB}^p means that the zero-coupon bond value rises and therefore corresponds to a decrease in the interest rate. This remark will be taken into account in what follows in order to analyze the interest rate risk exposure of the portfolio.

Up to a small model bias, the variables ε_{Stock} and ε_{ZCB} are, as previously observed, very near to a realization of Brownian motion increments of the stock and zero-coupon bonds. We will assume in what follows that, accordingly to the hypothesis H0, the vector $(\varepsilon^{S}_{Stock}, \varepsilon^{S}_{ZCB})$ is Gaussian and with standard normal marginal distributions.

After this treatment, it becomes possible to locate in the Cartesian plane the set of simulations in the first period. Each primary simulation corresponds, in fact, to a point with stock risk factor (respectively zero-coupon bond risk factor) in the abscissa (respectively in the ordinate). The following graph represents the set of points associated to all the simulations in the first period:

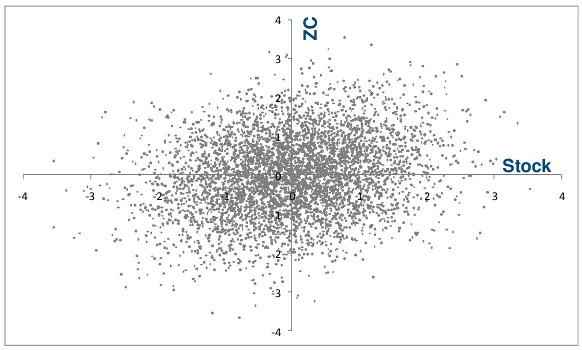


Figure 5: points associated to primary simulations

4.3. Execution region definition

We have seen previously that, with a thorough analysis of the results of an NS projection, and provided that the risk factors are carefully chosen, the worst case situations in terms of solvency correspond to the couple of factors situated in the outer area of the set of points. In the following graph, the « triangle » points represent the 50 worst values of equity at the end of the first period:

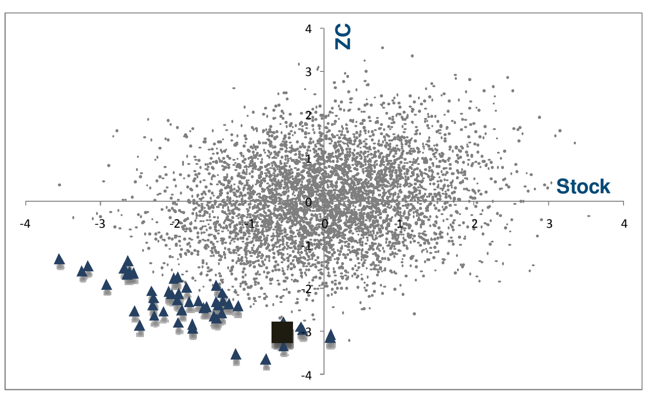


Figure 6 : location of the 50 worst case scenarios (triangles)

It then seems wise to carry on the primary projection starting with the points that are the farthest from the center and then to proceed gradually inwards.
The following graph confirms this implementation by putting in evidence the structure by percentile "contour lines" of the equity distribution:

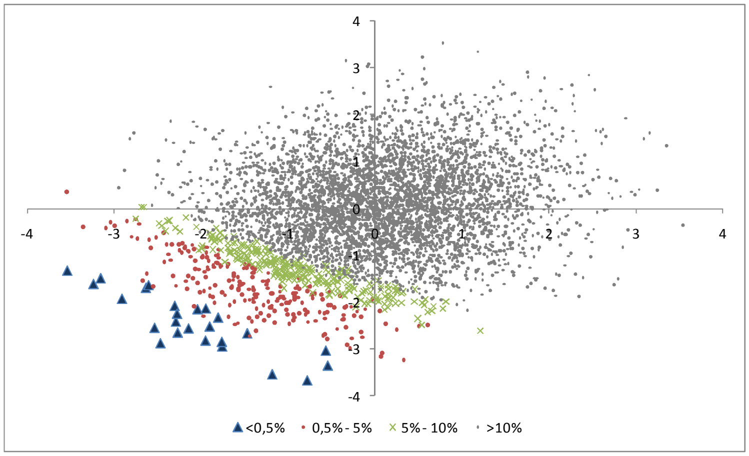


Figure 7 : contour regions by percentile thresholds

Figure 7 shows that the farther a point is from the "central conditions", the lesser the corresponding equity is (granted that this point is located in the "risk" zones).

The vector $Z = (\varepsilon_{Stock}, \varepsilon_{ZCB})$ is assumed to be Gaussian; the worst case situations correspond therefore to the factor couples for which the density f_Z of this vector is small.
Remember the Z distribution density:

$$f_Z(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}}.$$

Therefore, to go through the set of primary simulations from the worst case situations to the "central" situations becomes the same as considering successively the points of increasing density.

In order to do so let's introduce the contour line k for the density f_Z.It is composed of the points (x,y) such that:

$$f_Z(x,y) = k$$

$$\Leftrightarrow k = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}}$$

$$\Leftrightarrow\ x^2 + y^2 - 2\rho xy = -2(1-\rho^2)ln\left(2\pi k\sqrt{1-\rho^2}\right).$$

Consider the norm: $\|(x,y)\| \triangleq \sqrt{x^2+y^2-2\rho xy}$.

The higher the norm $\|(x,y)\|$ is, the lesser the point density is; in other terms the less solvable situations correspond to the points with higher norm.

In what follows we will refer to the following set as execution region of level h:
$F_h = \{(x,y)/ : \|(x,y)\| \geq h\}$.

Set F_h is therefore complementary to the open sphere centered upon the origin and with radius h, according to the topology induced by the norm $\|.\|$.

The border of the execution region F_h is an ellipse equal to the union of the following two arcs:

$$A_1^h = \left\{(x,y)/: y = \rho x + \sqrt{h^2 + \rho^2 x^2 - x^2}\right\},$$

and,

$$A_2^h = \left\{(x,y)/: y = \rho x - \sqrt{h^2 + \rho^2 x^2 - x^2}\right\}.$$

so: $\partial F_h = A_1^h \cup A_2^h$.

Fundamental Remark: to go through the set of primary simulations from the worst case scenarios to the "central" situations is equivalent to consider successively the regions of execution F_h for a decreasing level h.

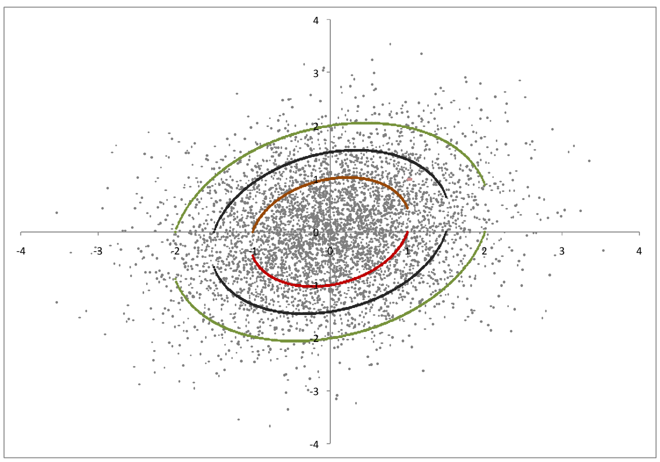


Figure 8 : F_h **border profile for** ***h=1 (red)***, ***h=1.5 (black)*** **and** ***h=2 (green)***.

4.4. The algorithm description

Remember that the goal of an NS projection is to estimate the 0,5% percentile of the equity distribution in a year. In order to do so it is enough to identify the $N = [0{,}5\% \times P]$ worst values of the sample $\left(FP_1^p\right)_{p=1,\dots,P}$.

The "NS-accelerator" automaton principle is supported on a **sorting by priority of the executions of the primary simulation starting from the worst cases**. The stabilization of the N worst values of equity from an iteration to the next ends the execution of the algorithm.

By executing the primary simulations from the largest to the smallest norm, the value of the projected equity becomes, after a certain number of simulations, consistently superior to the $N = [0{,}5\% \times P]$ worst values that we are looking for. In fact, as we introduce simulations they gradually become "central situations" which have equity fair values which are higher to those in the tail of the distribution.

The following graph represents, for the practical case considered in the study, the 25 primary simulations that are the most adverse and the border of the execution region F_h with a level $h = 2{,}5$.

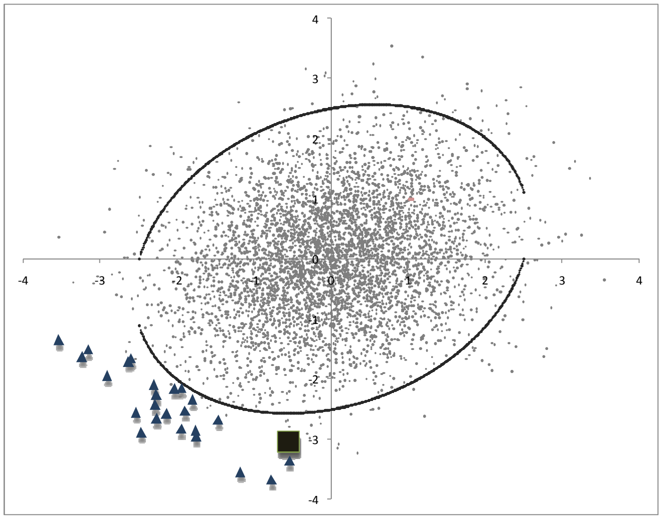


Figure 9 : Border of the execution region F_h with h=2.5 and the most adverse simulations

We can infer from Figure 9 that the primary simulations associated to the points with a norm less than 2.5 will lead to en equity fair value superior to the 25 worst values of $\left(E_1^p\right)_{p=1,\ldots,P}$.

It is on this principle that the NS-accelerator is based. The algorithm is as follows:

- ***Iteration 1*** : establishing the h_1 level such that there are M points[7] in the exterior of F_{h_1}
  - Equity calculation E_1 for each of the primary simulations associated to the elements of F_{h_1},
  - Sort by increasing order of the preceding values and storage of the $N = [0{,}5\% \times P]$ worst values in the vector V_1.
- ***Iteration 2*** : establishing the $h_2 < h_1$ level such that there are M supplementary points in the exterior of F_{h_2} (we then have $2M$ points : M points from the first iteration and M new points)
  - Determine the Equity E_1 for each of the primary simulation associated to these M new points,
  - Sort by increasing order the $2M$ obtained values and storage of the N worst values in the vector V_2 ,
  - If $V_1 = V_2$ -> **stop the algorithm**, else continue.

…

- ***Iteration i*** : establishing the $h_i < h_{i-1}$ level such that there are M supplementary points in the exterior of F_{h_2}

[7] It is logically necessary to have $M \geq N$. To avoid any "pitfall" in the algorithm we suggest to use enough points in each iteration, by taking for example : $M = 4N$.

- Determine the Equity E_1 for each of the primary simulation associated to these M new points,
- Sort by increasing order the $i \times M$ obtained values and storage of the N worst values in the vector V_i ,
- If $V_{i-1} = V_i$ -> **stop the algorithm**, else continue.

…

Remark: the number of iteration that the algorithm should execute is not fixed a priori. In fact, it is endogenous to the automaton. In the worst case, if the algorithm does not converge, the set of all primary simulations will be carried out.

4.5. Method advantages and possible improvements

4.5.1. Method advantages

- The most important advantage is an important reduction of the calculation time: the algorithm converges in about 300 to 500 primary simulations against 5000 simulations for an exhaustive NS.
  This scale change in the level of the calculation time allows to apply an NS methodology in companies having a large number of portfolios. It is in fact necessary for these companies to carry out an NS projection by portfolio (due to the profit sharing) and in such a case the standard NS methodology seems hard to implement[8].

- The NS-accelerator permits to incorporate a large number of risks (stock, interest rate, inflation, credit, mortality...). It is enough to come down to the Gaussian risk factors, which is usually natural according to the structure of the different models most used in the marked.

  To generalize the method into n dimensions, we consider the density of the Gaussian vector $(X_1, \dots, X_n)$ composed of the n risk factors:

$$h(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{det(V)}} exp\left\{-\frac{1}{2}.x'.V^{-1}.x\right\},$$

  where Vrepresents the variance-covariance matrix of this vector.
  The execution region F_h at an h level is defined in the same manner as it was done for 2 dimensions: $F_h = \{(x_1, \dots, x_n)/: \|(x_1, \dots, x_n)\| \geq h\}$ by choosing the norm $\|(x_1, \dots, x_n)\| = \sqrt{x'.V^{-1}.x}$.

  Next we will present the worst case situations in a model with three risks: (stock, interest rate and mortality):

[8] The complexity becomes $Number\ of\ porfolios \times P \times S$.

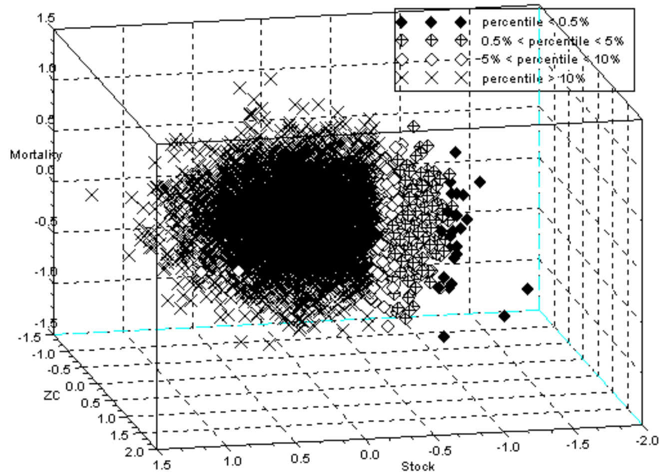


Figure 10 : risk factors and adverse situations in 3 dimensions (stock, interest rate et mortality)

- The method is not region-centered: for example at each iteration we have introduced a set of scenarios in which where included some scenarios with a high and others with a low interest rate. All possible adverse combinations are considered and no strong a priori assumption is made on the region that is less solvable; it is the accelerator that finds this zone automatically. This accelerator property allows to take into account the diversification effects that result from the consolidation of portfolios exposed to the increase / decrease of interest rate, of longevity / catastrophic mortality...

- The possibility to "locate" the "less solvable" region leads to a keen analysis of the nature of the company's risks and on the way to manage it.

4.5.2. Possible improvements

We will present in this section two variations of the accelerator that allow to reduce in a substantial manner the number of primary simulations. By combining them, the number of primary simulations is divided in general by a factor of 25 (200 simulations are then necessary to arrive to the same result than a standard NS based on 5000 primary simulations).

- In the case in which the company desires to establish the economic capital at a consolidated level without a capital allocation at an infra level (entity, business line...) it is possible to increase the performance of the NS-accelerator by introducing a region of "no execution".
  For example, a primary scenario in which the stock has a very good performance will normally lead, without considering the interest rate level, to a situation that is not adverse enough to impact the percentile estimation at a 0,5%. By introducing a "non execution region" which is re-evaluated at each of the NS-accelerator iterations, it becomes possible to substantially reduce the number of required simulations.

- The inclusion of a large number of factors can slow down the accelerator. In fact, if for example in the case of a model exposed to stock risk, interest rate risk and inflation, the worst case situations are explained only by the stock / interest rate shocks, the information supplied by the inflation risk will only reduce the speed of the algorithm. In order to deal with this difficulty, we introduce in the acceleration the notion of sensitivity to each factor. In order to do so, it is enough to modify the risk factors and the norm that defines the execution region.

  Denote $Z = (X_1, \dots, X_n)$ the Gaussian vector of risk factors and assume that the marginal distributions are standard normal. Denote $V^{1/2}$ the Cholesky transformation matrix of the variance-covariance matrix Z, we can define a new Gaussian vector $Y = (Y_1, \dots, Y_n)$ of risk factors, where the marginal distributions are independent standard normal, in the following manner:

  $Y = V^{1/2}.Z$.

  Each factor Y_i corresponds to the « pure effect » of the risk i.

  The norm used by the acceleration in this case is: $\|(y_1, \dots, y_n)\| = \sqrt{y_1^2 + \cdots + y_n^2}$.

  From a measure of the sensibility s_i to the factor y_i of the equity value in 1 year, we propose the following adjustment: $\|(y_1, \dots, y_n)\|^{ajust} = \sqrt{s_1.y_1^2 + \cdots + s_n.y_n^2}$.

  In the case of our example, the fact that the equity value in a year is less sensible to the inflation than to the other factors leads us to under-weight the inflation. This treatment will then allow us not to alter the NS-accelerator convergence.

- At last, it could be convenient to consider some more scenarios than the 0,5% worst in order to best estimate the desired Value-at-Risk.

5. Application on a standard savings portfolio

We will elaborate in this section on the implementation of the NS-accelerator on a typical company.

5.1.Portfolio and model description

The portfolio that we have considered in this study is a savings portfolio with a zero minimum guaranteed rate. We have projected this portfolio using Milliman's internal model. This model carries out ALM stochastic projections and does equity calculations over one year. These projections allow to model the profit sharing mechanism and the dynamic lapses behaviors of the policyholders when the distributed rates of return of the company seem not to be enough in comparison to other rates of return in the market.

We have only considered in this study the stock and interest rate risks. The economic scenario tables implemented were calibrated to 31/12/2008 market conditions

5.2.Results obtained from an exhaustive NS projection

In order to validate the results obtained from the NS-accelerator, we have carried out an exhaustive NS projection with 5000 primary simulations and 500 risk-neutral secondary simulations.

The empirical equity distribution that we obtained is as follows:

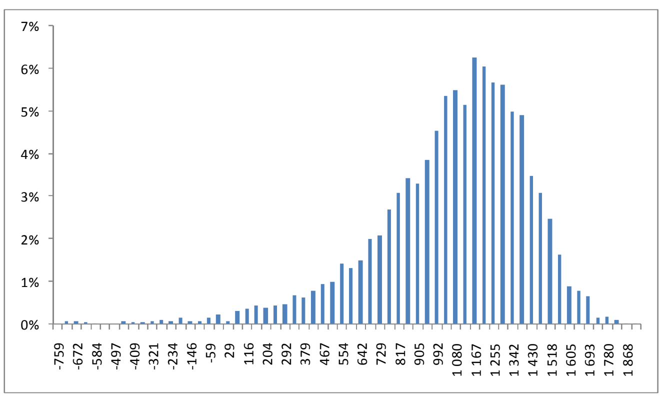


Figure 11 : Equity distribution at *t=1*

The following table represents some descriptive statistics of this distribution:

Mean	1 040
Median	1 097
Standard deviation	359
Kurtosis	2,7
Skewness	-1,2

Tableau 1 : descriptive statistics of the E_1 distribution

Our projection is supported on 5000 primary simulations, the 0,5% percentile of the equity in a year has been estimated by the 25th worst value of the empirical distribution. Therefore we have:

$$\widehat{q_{0,5\%}(E_1)} = E_1^{(25)} = -343{,}8.$$

The present equity level does not allow to ensure the solvency of the company in 1 year, it is thus necessary to supply additional capital. The one year interest rate in a year of the swap yield curve at 31/12/2008 is of 2,6% therefore the surplus S of capital to supply at $t = 0$ is:

$$S = \frac{343{,}8}{1+2{,}6\%} = 335{,}0.$$

We have extracted in each primary simulation the risk factors of the applied "real-world" table. The following table contains the 25 worst equity values that are compared with the values established by the accelerator after its convergence:

Position	Stock factors	ZC factors	E_1	Norm
1	-2,5	-2,9	-1 153	3,3
2	-2,5	-2,5	-1 099	3,2
3	-1,9	-2,8	-883	3,1
4	-2,3	-2,7	-843	3,1
5	-1,8	-2,9	-837	3,1
6	-3,2	-1,6	-758	3,3
7	-0,8	-3,7	-744	3,6
8	-1,2	-3,5	-724	3,5
9	-2,9	-1,9	-714	3,1
10	-1,8	-2,8	-712	3,0
11	-2,3	-2,4	-696	2,9
12	-2,1	-2,6	-673	3,0
13	-3,5	-1,3	-653	3,5
14	-1,9	-2,5	-647	2,8
15	-3,2	-1,5	-614	3,2
16	-2,3	-2,1	-487	2,7
17	-2,3	-2,3	-466	2,8
18	-0,5	-3,4	-462	3,3
19	-1,8	-2,3	-458	2,6
20	-1,5	-2,7	-438	2,7
21	-2,0	-2,2	-423	2,6
22	-2,6	-1,7	-388	2,8
23	-1,9	-2,1	-377	2,6
24	-2,6	-1,6	-350	2,8
25	-0,6	-3,0	-344	3,0

Table 2 : the 25 worst FP_1 values, the risk factors and the norms of the associated primary simulations

The following graph represents the couple of risk factors associated to the different levels of percentile distribution of E_1 :

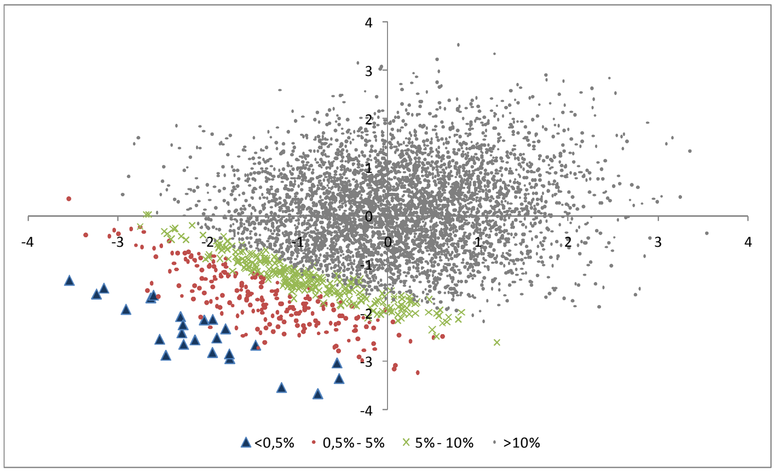


Figure 12 : risk factor couples by percentile level of the E_1 distribution

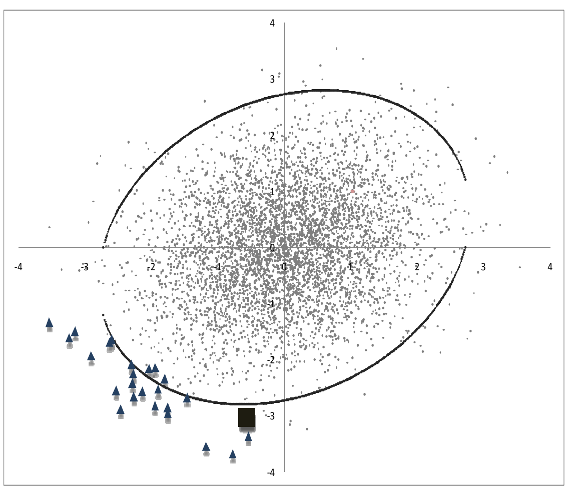
This graph guarantees to a certain level the NS-accelerator convergence. We observe in fact that the worst E_1 values are located in the outer areas of the set of points.

5.3.Implementation of the NS-accelerator

The NS-accelerator has carried out 300 primary simulations in order to converge. The algorithm has been set-up in order to execute 100 new points at each step ($M = 100$)[9].

Next we will present the results that we obtained at each iteration.

Iteration 1 - $h_1 = 2.73$



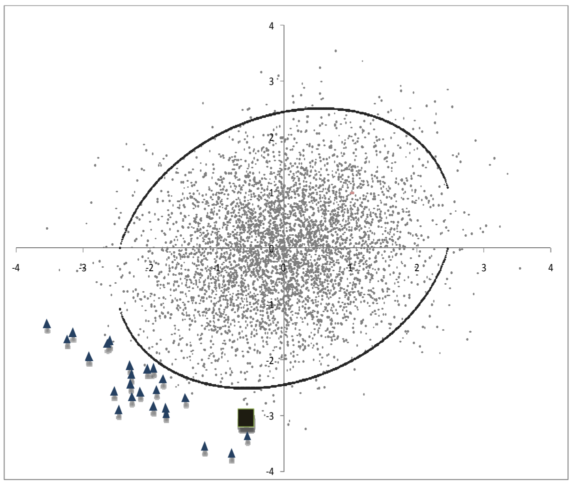
Position	E_1
1	-1153
2	-1099
3	-883
4	-843
5	-837
6	-758
7	-744
8	-724
9	-714
10	-712
11	-696
12	-673
13	-653
14	-647
15	-614
16	-487
17	-466
18	-462
19	-438
20	-388
21	-350
22	-344
23	-339
24	-295
25	-191

Table 3 : results for the first iteration

[9] $M = 4 \times [0{,}5\% \times 5000]$.

The previous graph allows us to observe the execution region border for a threshold of $h_1 = 2{,}73$ on the set of all points. The worst 24 values of E_1 established by the exhaustive NS correspond to the "triangle" points and the couple related to $E_1^{(25)}$ is represented by a square.
We observe that the first iteration does not allow to establish the 25 worst values since there are some "triangle" points that are not in the execution region

Iteration 2 - $h_2 = 2{,}45$

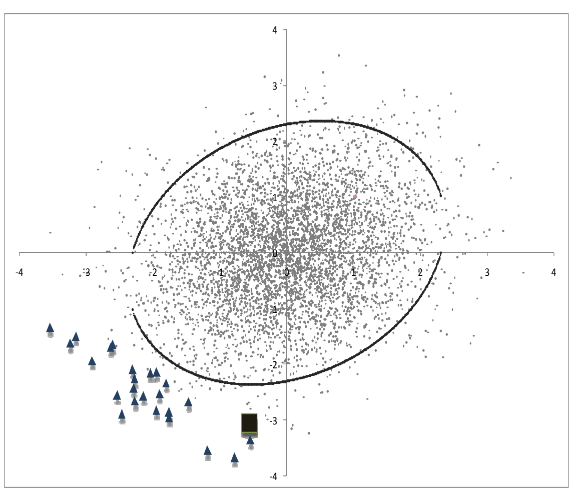


Position	E_1
1	-1153
2	-1099
3	-883
4	-843
5	-837
6	-758
7	-744
8	-724
9	-714
10	-712
11	-696
12	-673
13	-653
14	-647
15	-614
16	-487
17	-466
18	-462
19	-458
20	-438
21	-423
22	-388
23	-377
24	-350
25	-344

Table 4 : Second iteration results

The algorithm must carry-out a supplementary iteration since the values found in the second iteration are different from those in the first iteration. However, we observe that the entire set of "triangle" points is now in the execution region for a threshold of $h_2 = 2{,}45$. Therefore the worst values established here correspond to those of an exhaustive NS (refer to the table 4). A supplementary iteration is still necessary to arrive to the end of the algorithm.

Iteration 3 - $h_3 = 2{,}30$



Position	E_1
1	-1153
2	-1099
3	-883
4	-843
5	-837
6	-758
7	-744
8	-724
9	-714
10	-712
11	-696
12	-673
13	-653
14	-647
15	-614
16	-487
17	-466
18	-462
19	-458
20	-438
21	-423
22	-388
23	-377
24	-350
25	-344

Table 5 : Third iteration results

The 25 worst values found in this step are the same as those established before, consequently the algorithm then stops after the third iteration.

Remark: with a set-up of $M = 20$ the NS-accelerator converges also to the same values as those of the exhaustive NS and only 200 primary simulations are carried out in this case. However, as previously highlighted, M should be large enough to avoid any “pitfall” in the algorithm execution.

6. Conclusion

We have seen in this article that the economic capital definition, from a Solvency II perspective, requires to be able to dispose of the equity distribution at the end of the first year. Therefore, for a life insurance portfolio, the most adequate methodology in order to obtain this distribution is the NS approach. In fact, for such portfolios, the asset is normally valued though the application of Monte Carlo techniques due to the profit sharing mechanisms and the dynamic lapse mechanisms.

However the number of simulations carried out in an NS projection leads to significant execution times to a point that this can jeopardize the possibility to implement them in some insurance companies.

The NS acceleration permits to keep the advantages of the NS method while granting a significant reduction in the number of simulation with respect to a standard NS projection. The implementation of an NS calculation thus becomes conceivable no matter the size and the structure of the company considered.

This automaton is based on the localization of the situations that are the more adverse between the risk factors that bear the risk intensity of each primary scenario. The underlying principle is to go through the set of primary simulations from those farther from the center to those in the center until the tail values stabilize.

As in any global risk analysis, the identification of principal risk sources and the choice of pertinent factors are essential in the implementation of this algorithm. If this preliminary step could not be completed correctly, an additional step would be necessary to study the risk of trapping the algorithm, or of non-selection of most adverse scenarios

The methods used in this article are not supposed to be the ones that model the risks in the most appropriate manner (in particular for the longer management horizons), but they are often used by the practitioners and therefore provide a good test of the efficiency of our method.

We have fine tuned some variations of our algorithm. These variations allow to reduce in an even more significant manner the number of simulations required. It must be noted also that some research is being carried out presently. It includes the mortality risk or the changes in the stationary assumptions, among which are the correlation crises (refer to Biard et al. (2008)). These studies will be the subject of future publications.

APPENDIX : The linear volatility HJM model

The Heath-Jarrow-Morton linear Gaussian model is a model of HJM type where the volatility of the zero-coupon bonds is deterministic; therefore the spot yield is a Gaussian diffusion (Jamshidian (1990), El Karoui (1992)).

The dynamic of a zero-coupon bond with maturity T under a historic probability *P* is:

$$\frac{dP(t,T)}{P(t,T)} = (r(t) - q(t).\sigma(t,T))dt + \sigma(t,T)dB_t,$$

where $(B_t)_t$ is a one dimensional Brownian motion under *P*, and σ is a bounded deterministic function of the C^1class with respect to the second variable. We assume that $\sigma(T,T) = 0$.

The process q corresponds to the risk premium; we assume that it is bounded.

<u>*Theorem:*</u>

The value of a zero-coupon bond is:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} exp\left[\int_0^t (\sigma(s,T) - \sigma(s,t))dB_s + \frac{1}{2}\int_0^t (\sigma^2(s,t) - \sigma^2(s,T))ds \right. \\ \left. - \int_0^t q(s).(\sigma(s,T) - \sigma(s,t))ds\right].$$

■

If the theorem is considered at $t = 1$ we have the following relation:

$$\frac{P(1,T)}{P(0,T)} = \frac{1}{P(0,1)} exp\left[\int_0^1 (\sigma(s,T) - \sigma(s,1))dB_s + \frac{1}{2}\int_0^1 (\sigma^2(s,1) - \sigma^2(s,T))ds \right. \\ \left. - \int_0^1 q(s).(\sigma(s,T) - \sigma(s,1))ds\right].$$

If we take a linear volatility structure $\sigma(t,T) = \sigma.(T-t)$, we have that:

$$\frac{P(1,T)}{P(0,T)} = \frac{1}{P(0,1)} exp\left[\int_0^1 (\sigma.(T-s) - \sigma.(1-s))dB_s + \frac{1}{2}\int_0^1 (\sigma^2.(1-s)^2 - \sigma^2.(T-s)^2)ds \right. \\ \left. - \int_0^1 q(s).(\sigma.(T-s) - \sigma.(1-s))ds\right]$$

$$= \frac{1}{P(0,1)} exp\left[\sigma.(T-1).B_1 + \frac{1}{2}\int_0^1 (\sigma^2.(1-s)^2 - \sigma^2.(T-s)^2)ds \right. \\ \left. - \int_0^1 q(s).(\sigma.(T-s) - \sigma.(1-s))ds\right].$$

Let:

$$\mu_T^P = \frac{1}{2}\int_0^1 (\sigma^2.(1-s)^2 - \sigma^2.(T-s)^2)ds - \int_0^1 q(s).(\sigma.(T-s) - \sigma.(1-s))ds - ln\big(P(0,1)\big),$$

And,

$$\sigma_T^P = \sigma.(T-1),$$

Therefore we obtain the desired structure for the risk extraction of the zero-coupon bond:

$P(1,T) = P(0,T)e^{\mu_T^P + \sigma_T^P.Z^{ZC}}$ with $Z^{ZC} \sim N(0,1)$.

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