Axioma Research Paper
No. 28

November 15, 2010

Finance, Statistics, Accounting, Optimization and some *Alignment* Problems

Quantitative equity portfolio management has evolved into an inter-disciplinary activity that draws expertise from fields of finance, statistics, accounting and optimization. Each one of these streams is a matured discipline in itself, having its own body of knowledge and operates under assumptions that are usually well-accepted within the respective community. However, when concepts from these diverse fields are applied in a common setting, there is bound to be friction between various assumptions which get magnified due to the use of an optimizer.

This paper focusses on problems that arise due to interaction between three principal players in any quantitative strategy, namely, the alpha model, the risk model and the constraints. We present a detailed investigation of these misalignment problems, survey some of their common sources, analyze and document their effects on the ex-post performance of optimized portfolios and finally conclude with a practical and effective remedy in the form of augmented risk models to circumvent them.

Finance, Statistics, Accounting, Optimization and some Alignment Problems

Anureet Saxena Robert A. Stubbs

November 15, 2010

1 Introduction

The relationship between finance and optimization can best be described as a divorced couple who are still in love. This is evident from the investment community's reaction to Markovitz' Mean-Variance Optimization (MVO) approach, proposed almost fifty years ago. Markovitz' seminal work (Markowitz (1952)) brought together concepts of risk, return, risk aversion and diversification - the holy grail of Modern Portfolio Theory - under a unifying framework of optimization. His discovery, however, divided the investment community into three camps.

The first camp rejected the very hypothesis of using quantitative techniques to solve capital allocation problems. The second camp approached his discovery with cautious optimism, analyzed portfolios constructed on the basis of his recommendations, soon realized that the resulting portfolios were unintuitive, unwieldy and thus impractical, and finally joined members of the first camp becoming ardent critics of the MVO approach. The third camp, primarily composed of academics, decided to perform a postmortem analysis of the MVO portfolios, understand the reasons for deviation of the ex-post performance from ex-ante targets and use their analysis to suggest enhancements to Markovitz' original approach. One of the fundamental contributions of the third camp was the development of factor models to capture the sources of systematic risk and characterize the key drivers of excess returns. Their discoveries over a course of several decades had a huge impact with factor models becoming industry standards for both alpha and risk model developers.

It is worth emphasizing that alpha and risk models have different mandates which affects their choice of specific factors in the construction of the factor models. While alpha hunting is essentially a forward looking activity with the goal of predicting excess returns within the next rebalancing horizon, risk prediction focusses on explaining cross-sectional variability of the returns process. Phrased in the jargon of statistics, the first moment of the equity returns process drives alpha modelers while the second moment is the focus of risk modelers. These differences in the ultimate goals inevitably introduce misalignment between the alpha and risk factors.

While alpha and risk models are indispensable components of any active strategy, there is a third component, namely the set of constraints, which plays an important role in determining the composition of the optimal portfolio. Most real-life quantitative strategies have constraints that model desirable characteristics of the optimal portfolio. While some of these constraints may be mandatory, for example a client's reluctance to invest in stocks that benefit from alcohol, tobacco or gambling activities on ethical grounds, other constraints are a result of best practices in practical portfolio management. Examples of the latter type of constraints include industry, sector and asset exposure constraints, turnover constraints, tax considerations, etc. These constraints play an important role in affecting the portfolio construction process, and their interaction with the alpha and risk model constituents naturally accrues significance.

To summarize, quantitative equity portfolio construction entails complex interaction between alpha factors, risk factors and constraints. Problems that arise due to mutual misalignment between these three entities are collectively referred to as Factor Alignment Problems (FAP) and constitute the emphasis of the current paper. Our key contributions are summarized below.

1. Varied opinions of alpha and risk model developers on common factors derived from financial statements data and technical analysis manifest themselves as misalignment between the alpha and risk factors. The presence of constraints has an exacerbating effect on the misalignment problem, and introduces another layer of complexity (Section 2).

2. Using an optimization tool to construct optimal holdings has an unintended effect of magnifying the aforementioned sources of misalignment. The optimizer cherry picks the aspects of the alpha models that it deems desirable when gauged on the yardstick of marginal contribution to systematic risk. Consequently, it overloads the portion of alpha which is uncorrelated with all the user risk factors (Section 3).

3. The optimizer is implicitly taking an aggressive bet on the assumption that being uncorrelated with all the user risk factors is tantamount to lacking systematic risk altogether. Our empirical results on a test-bed of real-life active portfolios clearly refute the validity of the stated assumption, and suggest the existence of systematic risk factors which are missing from the user risk model (Section 4).

4. We propose augmenting the user risk model with an additional auxiliary factor to account for the effect of the missing risk factors. The augmenting factor is constructed dynamically and takes a holistic view of the portfolio construction process involving the alpha model, the risk model and the constraints. We provide analytical evidence to attest the effectiveness of the proposed approach (Section 5).

2 Misalignment Problems: Sources

As discussed in the previous section, any quantitative strategy has three key components, namely, alphas, a risk model and the constraints. Each one of these constituents has a distinct role in the portfolio management process, and hence by implication, a different emphasis. The alpha construction process, for instance, has a forecasting emphasis and the merit of an alpha model is judged by its ability to predict excess returns aka IC. The risk model generation, on the other hand, is geared towards explaining cross-sectional variability in the realized returns. The merit of a risk model is judged by its ability to capture systematic risk factors and the correlation structure between their respective factor returns. To cast it in the jargon of statistics, both the alpha and risk model concern the stochastic nature of the return process with the former focused on the first moment while the latter concerned with the second moment. This disparity in their respective objectives naturally affects the factors of interest in their construction, and introduces a modicum of misalignment. In this section, we give several examples of this inherent misalignment. The following section discusses how these sources of misalignment, no matter how insignificant they may appear, get magnified within an optimization framework and impact the ex-post performance. We refer the reader to Robinson et al. (2009) for definition of accounting terms used in this paper.

We start our discussion with one of the most fundamental attributes of an equity security, namely its earning potential. The earnings yield (E/P) is easily one of the most widely cited valuation ratio and receives wide coverage in both the academic and industry press. There are two key differences in the way earnings yields are used in the construction of alpha and risk models.

First and foremost, being an accounting based metric E/P is subject to various kinds of adjustments as deemed necessary by its end users. These adjustments can arise due to use of different accounting principles (U.S. GAAP versus IFRS), different treatment of income sheet entries (non-recurring, unusual and infrequent items, non-operating gains/losses) or different views on accrual accounting artifacts such as depreciation, amortization etc. These kinds of adjustments are more popular in the alpha construction process and relatively less in the generation of risk models. The reason for this disparity stems from different objectives of the risk and alpha model. With its primary focus on explaining the cross-sectional variability of the return process, a risk model can often make do with ballpark estimates and gains little, if at all, from razor sharp estimation of accounting entries. Alpha models, on the other hand, are focused on exploiting return differentials between various assets and adjustments like the ones mentioned earlier do have a significant impact.

For instance, consider the recent announcement by Dell to restate its earnings in the first quarter of 2011 and take a $100 million charge to settle the federal investigation of accounting fraud. While alpha hunters are likely to perform detailed accrual analysis of Dell's I/B statements to assess the long-term consequences of this event and possibly revise their forward looking earnings estimate, the risk model providers are unlikely to process this event extensively except for incorporating the effects that are already reflected in the current market price of Dell. To put it differently, this is an uninteresting problem from a risk model perspective because it is likely to affect only the specific return of Dell; ironically, it is the same asset specific nature of the event that makes it a remarkably attractive development from an alpha model perspective. It offers active managers the unique opportunity to harvest truly asset-specific returns without taking excessive exposure to any systematic risk factor. In other words, alpha and risk modelers have different beliefs about the possible impact, or lack thereof, of various economic events on their respective mandates and the misalignment between the alpha and risk factors is simply an inevitable manifestation of their diverse beliefs.

Second, alpha and risk model developers can at times take a completely different view on the issue of earnings potential altogether. For instance, some alpha construction techniques use alternative valuation metrics such as EBITDA/EV, FCFF/P or FCFE/P in lieu of E/P and for good reasons. For instance, small-cap growth companies in the technology sector often continue to have negative income and positive free cash flows in their early years. The negative income is usually the result of amortization of capital investments made at the inception of the company. For such businesses, free cash flow based valuation ratios are clearly better suited than earnings yield. Risk model providers, on the other hand, usually refrain from using the aforementioned alternatives to E/P. These different measurement choices of the same underlying fundamental metric, namely earnings potential, leads to misalignment between the alpha and risk factors.

Another interesting source of misalignment arises from the use of book-to-price (B/P) ratio. Roughly speaking, book value is the accounting profession's estimate of the company's value; it reflects what the company paid for the assets except intangible assets such as goodwill developed internally, but it includes goodwill of subsidiary companies acquired by purchase. This "cost basis" is then adjusted downward by depreciation and amortization in a highly stylized and rigid attempt to reflect the economic depreciation that actually befalls (most) assets. Off balance-sheet items are ignored. Finally the result is augmented by retained earnings. With book value reflecting such a melange, it is surprising that it has any explanatory power at all, but it does (Siegel (2003)). Its popularity can be gauged by the fact that it is the only factor used by S&P in differentiating value from growth stocks in each market capitalization segment.

Naturally, B/P is used extensively by both alpha and risk model developers. The spectrum of accounting issues discussed in context of E/P also apply to B/P The situation for the B/P factor, however, is far more complicated than E/P due to the presence of intangible assets, deffered taxes and goodwill. Recall that goodwill arises when the purchase price of net assets acquired exceeds the fair market value of other net assets acquired. In recent years, the growth of the service and knowledge sectors in the economy has changed the nature and extent of goodwill accounts created by M&A transactions. Firms increasingly derive their value from ability to create and exploit intellectual rights and human resources rather than from ownership of physical or financial assets. Compared to physical or finance assets, the estimate of intrinsic value of intellectual property and human resources is much more subjective, and essentially is a licence to accounting imagination (Dharan (1997)). Adjustments arising due to goodwill itself can create significant misalignment between the B/P factors employed in the alpha and risk models.

As mentioned earlier, the choice of factors for alpha and risk model construction is governed by different objectives. For instance, risk model developers often use adjusted R^2, root-mean square t-statistics or other measures of statistical significance to determine the suitability of an additional risk factor. There is usually tradeoff between the accuracy of risk model and the number of systematic risk factors that are employed. As a result of this tradeoff, some alpha factors might not make the cut during the risk factor selection process thereby further aggravating the misalignment problem. For instance, consider a scenario where a risk model developer has B/P and E/P as part of his suite of risk factors, and is evaluating the suitability of the S/P factor. It is possible that S/P has little marginal explanatory power beyond what is already captured by B/P and E/P, and he might decide to not include S/P as part of the risk model. If such a risk model is used in conjunction with alphas derived from the S/P factor, then the misalignment between the alpha and risk factors is inevitable.

Besides these factors which are based on financial statements analysis, misalignment can also arise due to different formulation of technical factors such as short-term and medium-term momentum. These momentum based factors have been extremely popular in the past fifteen years although they have come under attack since the 2008 financial crisis. It is worth noticing that there is no consensus on what constitutes an ideal momentum factor nor any firm theoretical guidance on how to choose one. Every alpha modeler has his customized view of these factors, and misalignment with a similar factor used in risk model generation is commonplace (Lee and Stefek (2008)).

Until now we have discussed misalignment issues between the alpha and risk factors. Next, we move our focus to the role of constraints in contributing to the misalignment problems. Consider the following constrained MVO model:

$$\begin{array}{ll} \max & \alpha^{\mathrm{T}} h - \frac{\lambda}{2} h^{\mathrm{T}} Q h \\ s.t & \\ & \qquad Ah \leq b \ , \end{array}$$

where α denotes the vector of expected returns, Q denotes the asset-asset covariance matrix, λ represents the risk aversion parameter and $Ah \leq b$ represents the constraints that the portfolio is required to satisfy. While some of these constraints, such as the sector/industry exposure constraints, are mandated by the client's investment policy statement (IPS), others such as turnover constraint are used to model the operational aspects of portfolio management. From theory of convex optimization and first order optimality conditions, we know that there exist $\pi \geq 0$ such that the optimal solution to the above problem is identical to the optimal solution to the following unconstrained MVO model:

$$\max \quad \left(\alpha - A^{\mathrm{T}} \pi\right)^{T} h - \frac{\lambda}{2} h^{\mathrm{T}} Q h \ .$$

The vector $\gamma = \alpha - A^{\mathrm{T}} h$ is referred to as implied alpha in the rest of this paper. Note that implied alpha is derived by tilting the α in the direction of binding constraints. In other words, implied alpha captures the effect of constraints in determining the optimal holdings and acts as *de facto* alpha for the constrained MVO model. As an immediate consequence, it follows that the extent to which implied alpha is spanned by the user risk factors will have a direct bearing on the composition of the optimal portfolio. We give two examples to illustrate this effect.

First, consider the effect of asset bound constraint, the most frequently occurring constraint in quantitative strategies. Most portfolio managers have explicit constraints that prohibit them from taking concentrated bets on a single asset. These constraints may result due to legal requirements or may represent the inherent nature of the strategy. An example of the legal requirement is ERISA (Employment Retirement Income Security Act) provision that disallows more than 10% investment of the defined benefit pension funds in a single asset. Similarly, strategies that closely track a given benchmark often impose tight bounds on individual active asset exposures. It is worth noticing that these constraints are usually binding for a large subset of the asset universe, and are responsible for the deviation of the implied alpha from alpha. More importantly, the degree of deviation is determined dynamically by the first order optimality conditions, and can introduce misalignment between γ and risk factors even when the alphas are spanned by the risk factors. In other words, misalignment arising due to asset bound constraints is an unavoidable feature of constrained MVO models which cannot be eliminated even if the the alpha and risk factors are in complete consonance. We refer the reader to Jagannathan and Ma (2003) for further discussion on the impact of asset bound constraints on the composition of optimal portfolios.

Second, consider the effect of a constraint that limits the exposure of the portfolio to illiquid assets. There are several ways to model the liquidity of assets, and different choices made during the construction of a "liquidity" risk factor and a liquidity constraint can introduce misalignment in the model. For instance, the liquidity risk factor could be determined using the daily trading volume of the respective assets while the coefficients used in the liquidity constraint could be obtained as a function of the average bid-ask spreads.

To summarize, misalignment is an integral feature of most quantitative strategies and arises due to the complex interaction between the alpha factors, risk factors and constraints. While a small amount of misalignment is innocuous in itself, when present in an optimization framework it can lead to some unexpected consequences. The section that follows dwells on some of these undesirable effects of misalignment.

3 Misalignment Problems: Effects

For ease of discussion we focus on the following unconstrained MVO model:

$$\max \quad \alpha^{\mathrm{T}} h - \tfrac{\lambda}{2} h^{\mathrm{T}} Q h \ ,$$

where $Q = X\Omega X^{\mathrm{T}} + \sigma_u^2 I$ is the asset-asset covariance matrix, X is the factor exposure matrix, Ω is the factor-factor covariance matrix, σ_u is the asset specific risk which is assumed to be constant across the asset universe, and λ is the risk aversion parameter; to simplify the notation we assume that $\lambda = 2$ and $X^{\mathrm{T}}X = I$. All the results presented in this section can be generalized to constrained MVO models by replacing alpha (α) by implied alpha (γ) as discussed in Section 2.

Let α_X denote the projection of α to the vector space spanned by columns of X and let $\alpha_\perp = \alpha - \alpha_X$ denote the residual component which is orthogonal to columns of X. There is a subtle connection between the orthogonal component $\alpha_\perp$ of α and the notion of misalignment discussed at length in the previous section. Specifically, if there is no source of misalignment then α is completely spanned by the risk factors and $\alpha_\perp = 0$. On the other hand, if the alpha factors are not completely spanned by the risk factors then $\alpha_\perp$ captures the amount of misalignment between the two sets of factors. In view of this discussion, we define *misalignment coefficient* (MC) of alpha to be,

$$MC(\alpha) = \frac{\|\alpha_\perp\|^2}{\|\alpha\|^2} \ .$$

Thus the higher the misalignment coefficient the higher is the degree of misalignment between the alpha and risk factors. We define h_X, $h_\perp$ and $MC(h)$ similarly. Next we discuss the impact of using an optimizer on the relative MC of α and h.

First, consider the case when $\Omega = 0$ i.e. there is no systematic component of risk. In this case, the optimal portfolio h is given by $h = \frac{1}{\sigma_u^2}\alpha$. Thus $h_X = \frac{1}{\sigma_u^2}\alpha_X$, $h_\perp = \frac{1}{\sigma_u^2}\alpha_\perp$, and $MC(h) = MC(\alpha)$. The optimal portfolio in this case is merely a reflection of the alpha, has the same relative composition of the spanned and orthogonal component, and hence the same misalignment coefficient as α. Phrased differently, it follows that in the absence of systematic risk factors, the optimizer is indifferent to the alignment between the alpha and risk factors, or lack thereof.

This situation undergoes a drastic change when Ω is different from zero, the more commonly occurring scenario. In this case, it can be shown that

$$\begin{aligned} h_X &= \frac{1}{\sigma_u^2}\alpha_X - \frac{1}{\sigma_u^2} X M^{-1} X^{\mathrm{T}} \alpha_X \\ h_\perp &= \frac{1}{\sigma_u^2}\alpha_\perp \end{aligned}$$

where $M = \sigma_u^2 \Omega^{-1} + I$. Note that while the orthogonal component $h_\perp$ of the portfolio is identical to the case when $\Omega = 0$, the spanned component undergoes some kind of diminutive transformation. To understand this disparity between the spanned and orthogonal component, recall that the optimizer perceives no systematic risk in $h_\perp$ since $X^{\mathrm{T}} h_\perp = 0$. Thus it exploits the systematic risk arbitrage between the spanned and orthogonal component thereby over-weighing the orthogonal component relative to the spanned component. The proposition that follows formalizes this over-weighing argument using the notion of misalignment coefficient.

Proposition 1.

$$MC(h) > MC(\alpha) \ .$$

Proof. Since $\|h_\perp\| = \frac{1}{\sigma_u^2}\|\alpha_\perp\|$, it suffices to show that $\|h_X\| < \frac{1}{\sigma_u^2}\|\alpha_X\|$. Note that,

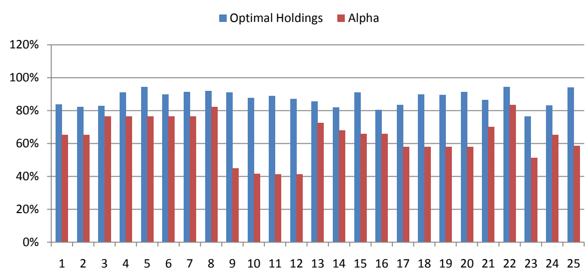
$$\begin{aligned} \sigma_u^4 \|h_X\|^2 &= \|(I - XM^{-1}X^{\mathrm{T}})\alpha_X\|^2 \\ &= \|X(I - M^{-1})X^{\mathrm{T}}\alpha\|^2 \\ &= \alpha^{\mathrm{T}} X \left(I - M^{-1}\right)^2 X^{\mathrm{T}} \alpha \\ &< \alpha^{\mathrm{T}} X X^{\mathrm{T}} \alpha \\ &= \|\alpha_X\|^2 \ . \end{aligned}$$

□

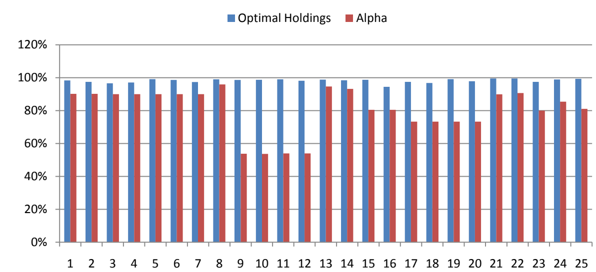
The above proposition shows that the optimizer has a magnifying effect on the misalignment between the alpha and risk factors. To give the reader better appreciation of the extent of this magnification effect we conducted the following experiment.

We constructed a test-bed of 25 backtests derived from real alphas and strategies used by Portfolio Managers (PM). This test-bed of backtests comprises a fairly heterogeneous collection of active investment strategies involving long-only, long-short short-extension and dollar-neutral portfolios rebalanced either daily, monthly, or quarterly; the number of periods in the backtests range from 50 to 170. Many of them have some form of factor neutrality constraints to hedge their exposure to style or industry factors. Each backtest was run twice, once with a cross-sectional fundamental risk model and once with a statistical risk model built using asymptotic principal components.

Subsequently, for each backtest we computed the average misalignment coefficient of the alpha and optimal holdings, as reported in Figure 1. The consistently higher value of MC for the optimal holdings as compared to that of the alphas is not surprising (see Proposition 1); what is indeed surprising is the magnitude of the difference between the two values. For some of the backtests, the MC of the optimal holdings was 100% greater than that of the corresponding alphas. On average, the MC of the optimal holdings was roughly 43% higher than that of the alphas for backtests that used the fundamental risk model; the same metric for backtests that used the statistical risk model was 26%.



(a) Fundamental Model



(b) Statistical Model

Figure 1: Misalignment Coefficient (MC)

It is worth emphasizing that misalignment coefficient is not a normative concept, instead, it is a descriptive statistic that captures the skewness in the composition of the optimal holdings or alphas. In other words, it cannot be used as the sole basis to determine the desirability, or lack thereof, of a given portfolio. MC becomes meaningful only when it is examined in light of the source of misalignment, and specifically the amount of residual systematic risk in the misalignment source which is not accounted for during the portfolio construction phase. We give two examples to illustrate this point.

First, consider a situation wherein the alpha model is based on the S/P factor whereas the risk model does not have a S/P factor even though it includes the E/P and B/P risk factors. In this case, the orthogonal component of alpha is defined to be the residual portion of S/P which is not captured by the existing risk factors such as E/P and B/P. It is possible that the mentioned residual component has systematic risk albeit not as significant as that of E/P or B/P factors. Since the optimizer fails to perceive any systematic risk in $\alpha_\perp$ it is likely to take excessive exposure to $\alpha_\perp$ thereby increasing $MC(h)$ relative to $MC(\alpha)$. In this case, the increase in the misalignment coefficient is indeed undesirable since it signifies the presence of latent systematic risk in the portfolio which goes undetected during the portfolio construction phase.

Second, consider the example of Dell's earnings restatement described in Section 2. Suppose an active manager of a large-cap growth fund uses TTM earnings yield E/P to define the alphas and also as a risk factor. Furthermore, in response to Dell's earnings restatement he decides to modify the exposure of his alphas to Dell but leaves the corresponding risk factor exposures unchanged. Such a scenario is especially likely to happen if the risk model is not developed in-house but purchased from a third party vendor making it prohibitively expensive to re-calibrate the risk model after modifying the factor exposures. In this case, the orthogonal component of α is truly asset specific having exposure to exactly one asset, namely Dell, and thus has insignificant systematic risk. Consequently, the effect of misalignment between the alpha and risk factors is innocuous and unlikely to damage the ex-post performance of the optimal portfolio, the magnification of the MC notwithstanding.

Note that all of these arguments continue to hold true in the presence of constraints, provided that alpha is replaced by implied alpha. While this may appear to be a minor modification from a conceptual standpoint, it introduces a tremendous amount of complexity from the purpose of analysis. To see this, recall that implied alpha (γ) is not a "static" entity i.e. implied alpha cannot be determined apriori independently of the constraints in the model. Furthermore, γ depends on the optimal portfolio and parameters of the first order optimality conditions as discussed in the previous section. The proposition that follows discusses an interesting property of the orthogonal component of γ which makes it possible to generalize the results for the unconstrained MVO model to the constrained case; the proof of the proposition is straightforward and omitted for the sake of brevity.

Proposition 2.

$$\frac{1}{\|\gamma_\perp\|}\gamma_\perp \quad = \quad \frac{1}{\|h_\perp\|}h_\perp \ .$$

As a corollary to the above proposition it follows that we can use the orthogonal portion $h_\perp$ of the portfolio as a proxy to $\gamma_\perp$ to analyze and understand the sources of misalignment in a constrained MVO model. Note that in the absence of constraints, $\gamma = \alpha$ and the above proposition reduces to the familiar equation $\frac{1}{\|\alpha_\perp\|}\alpha_\perp = \frac{1}{\|h_\perp\|}h_\perp$ which we have already encountered before.

Next we discuss some of the unintended consequences of the factor alignment problem. One of the most conspicuous effect of FAP is the reduced accuracy of risk forecasts (see Saxena and Stubbs (2010a)). The deterioration in the quality of risk prediction is particularly acute when the orthogonal component of alpha has significant systematic risk. The mentioned case symbolizes the situation wherein the fund manager is trying to reap risk premium to a systematic risk factor which is not included in the risk model. For instance, consider using alphas derived from Pastor and Staumbaugh' liquidity factor along with a risk model defined using the Fama-French risk factors; the returns to the optimal portfolio from such a combination of alpha and risk model essentially represent the risk premium to the liquidity risk factor. Since the Fama-French model does not have a liquidity risk factor, the ex-ante risk prediction of the optimal portfolio will inevitably suffer from a downward bias.

Factor misalignment also introduces a disconnect between the techniques employed in the alpha construction phase, and how those alphas eventually get transformed into an optimal portfolio. For the sake of discussion consider an active PM who is using a weighted combination of two alpha signals to construct his alphas. One of these signals, say α_G, is identical to the growth factor present in his risk model while the second signal, say α_S, is uncorrelated with all the risk factors in his risk model. Based on inputs from his research team members, the PM concludes that α_G has roughly three times more strength than α_S and decides to use the following formula to define his alphas,

$$\alpha = \frac{3}{4}\alpha_G + \frac{1}{4}\alpha_S \ .$$

Note that the PM's intention is to tilt his holdings towards the growth factor thereby introducing a style bias in his active portfolio. Based on the results presented earlier and Proposition 1 we know that the optimizer views the two components of alpha, α_G and α_S, differently; it favors α_S over α_G for lack of exposure of α_S to risk factors in the PM's risk model. Thus, the optimal portfolio is very likely to have much higher exposure to α_S than α_G thereby nullifying the original intention of the PM.

To summarize, the optimizer cherry picks the aspects of the alpha models that it deems desirable when gauged on the yardstick of marginal contribution to systematic risk. In its endeavor to do so it pushes most of the optimal holdings into the vector space which is orthogonal to one defined by the user risk factors. In other words, the optimizer is taking an aggressive bet on the assumption that being uncorrelated with all the user risk factors is tantamount to lacking systematic risk altogether. As we discuss in the following section, this assumption turns out to be the Achilles' heel of optimized active portfolios.

4 Misalignment Problems: Analysis

In this section we present an empirical analysis of the factor alignment problem. Specifically, we show that despite being uncorrelated with all the risk factors in the user risk model, $h_\perp$ does have significant systematic risk and its cross-sectional explanatory power is comparable to that of an average fundamental or statistical risk factor. Next we describe the experimental setup used to establish the stated hypothesis.

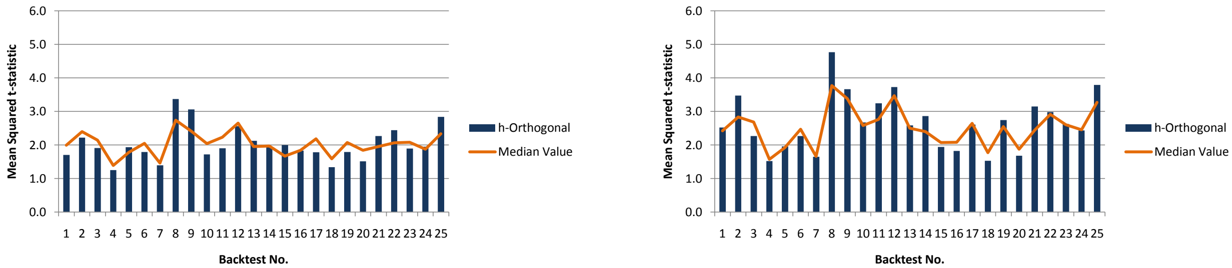
For each backtest in our test-bed we performed a weighted cross-sectional regression analysis on asset returns using all the regular risk factors in the risk model and $h_\perp$; we used the optimal portfolio holdings to weigh the regression model in each period (see Section 4 of Saxena and Stubbs (2010a) for more details). Subsequently, we performed a time series analysis on the factor returns of $h_\perp$ to determine the systematic risk in $h_\perp$ and also the statistical significance of $h_\perp$ viewed as an additional risk factor.

Figure 2 reports the root mean square (RMS) t-statistic corresponding to $h_\perp$ in the above experiment. For sake of comparison, we also report the median RMS t-statistic corresponding to regular factors in the respective risk model. Note that the RMS t-statistic value for $h_\perp$ is close to the 95% threshold value of 1.96 for several backtests, and often exceeds the median value corresponding to regular risk factors. This implies that not only is $h_\perp$ a statistically significant risk factor, its explanatory power is comparable to that of an average risk factor in the fundamental and statistical risk model. It is interesting to note that $h_\perp$ compares favorably with such a carefully chosen set of risk factors thereby corroborating its significance as a systematic risk factor.

Figure 3 reports the annualized standard deviation of factor returns of $\frac{1}{\|h_\perp\|}h_\perp$ which is a measure of the systematic risk in $h_\perp$. Note that $h_\perp$ has roughly $20-30\%$ annualized volatility which is commensurable with the volatility of an average risk factor in the fundamental risk model. These statistics clearly show that $h_\perp$ has systematic risk which is not correctly accounted for during the portfolio construction process.

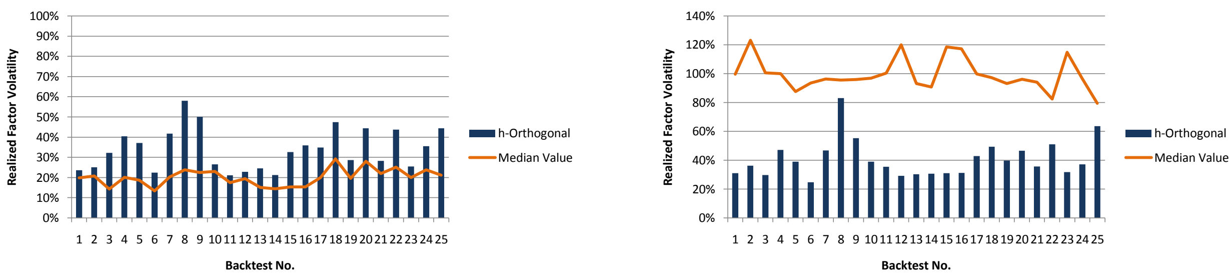
Our key findings until now can be summarized as follows.

1. There is an inevitable misalignment between the alpha factors, risk factors and constraints in any quantitative strategy (Section 2).



(a) Fundamental Model (b) Statistical Model

Figure 2: Root mean squared t-statistic



(a) Fundamental Model (b) Statistical Model

Figure 3: Realized factor volatility of $h_\perp$

2. During the construction of the optimal holdings, the optimizer is misled into believing that the orthogonal component $h_\perp$ of the optimal portfolio has no systematic risk and only idiosyncratic risk. As a result, it favors the orthogonal component relative to the spanned component resulting in skewed composition of the optimal portfolio (Section 3).

3. Empirical analysis shows that not only does $h_\perp$ have systematic risk, its cross-sectional explanatory power is comparable to an average factor in the fundamental and statistical risk model (Section 4).

4. The primary purpose of portfolio optimization is to create a portfolio having an optimal risk-adjusted expected return. If a portion of the risk in a portfolio is not accounted for, then the resulting risk-adjusted expected return cannot be optimal.

Having identified the crux of the problem, the next logical step is to educate the optimizer about the systematic risk in $h_\perp$ and assist it in correctly accounting for the systematic risk during the portfolio construction phase. The section that follows develops this line of thinking culminating with a practical and effective solution to the factor alignment problem. However, at this point we take a brief detour and ask ourselves another pertinent question, namely, why do risk models fail to capture the systematic risk in $h_\perp$?

The answer to this question lies in the statistical intricacies of risk model development, and deserves some attention. Recall that in any multi-variate linear regression model there is a tradeoff between explanatory power and model accuracy. In other words, one can increase the R^2 of a regression model by adding another risk factor but that increase comes at a cost, namely, higher standard errors in the factor returns computation. Risk model providers, being fully aware of this tradeoff, restrict the number of factors in their risk model to a handful few. For instance, Axioma's fundamental risk model has around 10 style factors and 68 industry factors which are derived from the GICS classification.

It is no coincidence that the academic community has devoted considerable resources in the past fifty years to develop multi-factor models for equity markets which have at most a dozen factors. Examples include the classical CAPM model (1 factor), Fama-French model (3 factors), Pastor and Staumbaugh extensions to Fama-French models (4 factors), macro-economic BIRR model (5 factors) etc[1]. It is worth observing that all of these models are developed with the intention to orderly explain an otherwise chaotic equity return process. However, none of these models, including those provided by commercial risk model vendors such as Axioma or MSCI-Barra, claim to capture "all" systematic risk factors. The malady of factor alignment problems lies in the optimizer's propensity to neutralize the exposure of the optimal portfolio to all systematic risk factors that are included in the risk model, and inadvertently load up on others which are omitted. This explains the significant amount of systematic risk in $h_\perp$.

To conclude, the optimizer plays the role of a devil's advocate exacerbating the mis-alignment between the alpha factors, risk factors and constraints, and thereby pushing the optimal holdings into a territory where the factor risk model is most likely to under-estimate systematic risk. The section that follows discusses the use of an augmented risk model in solving these problems.

5 Misalignment Problems: Solution

One of the key takeaways from previous two sections is that factor risk models include only a subset of all systematic risk factors that come to bear on the equity returns process. In other words, there are additional risk factors which are missing from the risk model and FAP arise when the optimizer aligns the portfolio with these missing factors. In light of these observations, we assume that the true asset-asset covariance matrix Q_T is given by

$$Q_T = Q + Z\Lambda Z^{\mathrm{T}} \ ,$$

where Z denotes the exposure matrix of the missing factors and Λ denotes the covariance between the factor returns of the missing factors. For the sake of brevity of notation, we assume that $Z^{\mathrm{T}}X = 0$ and $Z^{\mathrm{T}}Z = I$. Our goal in this section is to evaluate the undesirable effects of the missing factors on the ex-post performance of the optimized portfolios, and make recommendations to overcome them. Similar to the previous sections we mostly restrict our focus to the unconstrained MVO model, and briefly address the constrained case at the end of this section. We use the following utility function to evaluate the ex-post performance of

[1]The descriptions of these factor models can be found in any standard textbook on equity analysis.

an arbitrary portfolio h,

$$U_T(h) = \alpha^{\mathrm{T}} h - \frac{\lambda}{2} h^{\mathrm{T}} Q_T h \ .$$

We define $U(h) = \alpha^{\mathrm{T}} h - \frac{\lambda}{2} h^{\mathrm{T}} Q h$ to be the ex-ante estimate of the utility function. Note that if $\Lambda = 0$ then the ex-ante and ex-post performance of h are identical. However, if Z and Λ are different from 0 then $U_T(h) \leq U(h)$. Of course, the degree of ex-post inefficiency of h depends on how far $U_T(h)$ can deviate from $U(h)$. The proposition that follows sheds some light on this issue apropos optimal MVO portfolios. Let $\sigma_s(\alpha_\perp) = \sqrt{\frac{1}{\|\alpha_\perp\|^2} \alpha_\perp^{\mathrm{T}} Z \Lambda Z^{\mathrm{T}} \alpha_\perp}$ denote the systematic risk of $\frac{1}{\|\alpha_\perp\|} \alpha_\perp$.

Proposition 3. *Saxena and Stubbs (2010b) If h is the optimal portfolio to MVO, then*

$$U_T(h) \quad = \quad U(h) - \frac{\|\alpha_\perp\|^2}{2\lambda\sigma_u^2} \sigma_s^2(\alpha_\perp) \ .$$

Thus $\frac{\|\alpha_\perp\|^2}{2\lambda\sigma_u^2} \sigma_s^2(\alpha_\perp)$ is the marginal ex-post cost for inadequate accounting of the systematic risk in $\alpha_\perp$. Note that if $\sigma_s(\alpha_\perp) = 0$ then $U_T(h) = U(h)$. In other words, the ex-post utility function differs from the ex-ante target only when $\alpha_\perp$ has systematic risk. Furthermore, the deviation in the ex-post performance increases with $\sigma_s(\alpha_\perp)$.

Our solution to FAP is based on using an augmented risk model obtained by augmenting the user risk model with an additional factor $y = \frac{1}{\|\alpha_\perp\|} \alpha_\perp$ which is orthogonal and uncorrelated with all the existing risk factors. Specifically, we define

$$Q_y \quad = \quad Q + \nu y y^{\mathrm{T}} \ ,$$

where ν denotes the systematic variance of y, and we solve the following optimization problem to determine the optimal portfolio,

$$\max \quad \alpha^{\mathrm{T}} h - \tfrac{\lambda}{2} h^{\mathrm{T}} Q_y h \ (AugMVO) \ .$$

The theorem that follows compares the ex-post performance of the optimal solution to (MVO) and (AugMVO).

Proposition 4. *Saxena and Stubbs (2010b) If h denotes the optimal solution to (MVO) and h_y denotes the optimal solution to (AugMVO), then the following statements hold true.*

1. $U_T(h_y) = U_T(h) + \frac{\|\alpha_\perp\|^2}{2\lambda\sigma_u^2} \frac{\nu^2}{\sigma_u^2 + \nu}$.
2. $h_y^{\mathrm{T}} Q_T h_y = h_y^{\mathrm{T}} Q_y h_y$.

The above theorem has several interesting consequences that we discuss next.

First and foremost, it establishes the superior ex-post performance of h_y relative to h. In other words, optimal portfolios derived from the augmented risk model are guaranteed to have better ex-post performance than those derived from the original user risk model. This improvement in performance can be attributed to augmented risk model's property of correctly accounting for the systematic risk in $\alpha_\perp$. Second, $h_y^{\mathrm{T}} Q_T h_y = h_y^{\mathrm{T}} Q_y h_y$ implies that the ex-ante risk estimate of h_y exactly matches the ex-post realized risk. In other words, using the augmented risk model completely eliminates the downward bias in risk prediction, one of the most serious consequences of FAP.

Third, the technique of using the augmented risk model can be implemented without explicit access to the missing factors. Thus even though we used the missing factors Z to provide a theoretical structure to our analysis, the final solution to the FAP magically circumvents them altogether. To see this, note that there are only two additional inputs required for the implementation of the augmented risk model, namely the factor $y = \frac{1}{\|\alpha_\perp\|}\alpha_\perp$ and ν, the systematic variance of y. While y can be computed using standard techniques from linear algebra, ν can be estimated by using the historical returns to factor y. From a statistical standpoint, the augmented risk model thus represents a compromise between the user risk model that misses some of the systematic risk factors, namely Z, and the true risk model that accounts for all the systematic risk factors but may be unwieldy to use due to the large number of factors. By augmenting the user risk model with a single factor y, the augmented risk model optimally aggregates the effect of all the missing factors into one factor so as to give tangible improvement in the ex-post performance of the optimal portfolio.

All of these results can be generalized to the constrained case by replacing α by implied alpha (γ) as discussed in Section 2. The resulting augmenting factor $\frac{1}{\|\gamma_\perp\|}\gamma_\perp$, however, depends on the optimal portfolio and parameters of the first order optimality conditions. The problem of simultaneously choosing an optimal portfolio and an augmenting factor can be phrased as an equilibrium problem which in turn can be reformulated as a convex second order cone program (SOCP). The details of the mentioned reformulation are technical and outside the scope of this paper (see Saxena and Stubbs (2010b)).

6 Conclusion

Quantitative portfolio management has evolved into an inter-disciplinary activity that draws expertise from fields of finance, statistics, accounting and optimization. Each one of these streams is a matured discipline in itself, having its own body of knowledge and operates under assumptions that are usually well-accepted within the respective community. However, when concepts from these diverse fields are applied in a common setting there are bound to be frictions between various assumptions which get magnified due to the use of an optimizer.

For instance, risk model developers usually restrict the number of systematic risk factors in their model to limit estimation errors with the "assumption" that the missing factors have insignificant systematic risk. Alpha model developers focus on generating stock specific alphas and "assume" that the risk model will hedge away exposures to systematic risk factors. The optimizer examines both the alpha and risk model, and "assumes" that the residual portion of alpha that is not spanned by the risk factors included in the risk model has only idiosyncratic risk, and loads up on the residual portion. As a result, the optimizer inadvertently takes excessive exposure to systematic risk factors which were dropped during the risk model construction resulting in factor misalignment problems.

In this paper, we examined the sources of misalignment, analyzed and documented their effects, presented a detailed analysis culminating with a practical and effective remedy to the factor alignment problems. Unsurprisingly, our research on the FAP brings together elements from the fields of finance, statistics, accounting and optimization, and is based on careful understanding of how these fields interact with each other within the world of equity investments. For instance, the fundamental source of FAP is the different view alpha and risk model developers have on financial statements data, an *accounting* artifact. The tradeoff between model accuracy and the number of factors restrict the risk models from incorporating all sources of systematic risk, a *statistics* artifact. The combined effect of the misalignment between the alpha and risk factors, and limited representation of the systematic risk factors in the risk model results in hidden systematic risk in the orthogonal component of (implied) alpha which gets magnified and excessively represented during the construction of the optimal portfolio, an *optimization* artifact. All of these influences have an ominous bearing on the ex-post performance, create a downward bias in the ex-ante risk prediction and thus compromise the ex-post efficiency of the optimal portfolio, a *finance* artifact.

The augmented risk model approach strikes a careful balance between all of these influences, and in the process creates a practical and effective solution to the factor alignment problem. It not only corrects for risk underestimation bias of optimal portfolios but also pushes the ex-post efficient frontier upwards thereby empowering a PM to access portfolios that lie above the traditional risk-return frontier defined by the user risk model.

References

Bala G. Dharan. Magnitude of goodwill in U.S. corporate financial statements. In Stephen A. Zeff and Bala G. Dharan, editors, *Readings and Notes on Financial Accounting*. New York: McGraw-Hill, Inc., 1997.

Ravi Jagannathan and Tongshu Ma. Risk reduction in large portfolios: Why imposing the wrong constraints helps. *Journal of Finance*, 58:1651–1684, August 2003.

Jyh-Huei Lee and Dan Stefek. Do risk factors eat alphas? *The Journal of Portfolio Management*, 34(4):12–25, Summer 2008.

Harry Markowitz. Portfolio selection. *The Journal of Finance*, VII(1):77–91, March 1952.

T.R. Robinson, H. Greuning, E. Henry E., and M. Broihahn. In *International Financial Statement Analysis*. CFA Institute Investment Series, 2009.

Anureet Saxena and Robert A. Stubbs. Alpha alignment factor: A solution to the underestimation of risk for optimized active portfolios. Technical report, Axioma, Inc. Research Report #015, February 2010a.

Anureet Saxena and Robert A. Stubbs. Pushing frontiers (literally) using alpha alignment factor. Technical report, Axioma, Inc. Research Report #022, February 2010b.

Laurence B. Siegel. U.S. equity style indexes. In *Benchmarks and Investment Management*, chapter 8. The CFA Institute Research Foundation, 2003.

Axioma
Flexible is better.™

United States and Canada: 212-991-4500
Europe: +44 20 3170 8745
Asia: +852-8203-2790

New York Office
Axioma, Inc.
17 State Street
Suite 2550
New York, NY 10004

Phone: **212-991-4500**
Fax: **212-991-4539**

Atlanta Office
Axioma, Inc.
8800 Roswell Road
Building B, Suite 295
Atlanta, GA 30350

Phone: **678-672-5400**
Fax: **678-672-5401**

San Francisco Office
Axioma, Inc.
201 Mission Street
Suite #2230
San Francisco, CA 94105

Phone: **415-614-4170**
Fax: **415-614-4169**

London Office
Axioma, (UK) Ltd.
288 Bishopsgate
London EC2M 4QP

Phone: **+44 20 3170 8745**

Hong Kong Office
Axioma, (HK) Ltd.
Unit C, 17/F
30 Queen's Road Central
Hong Kong

Phone: **+852-8203-2790**
Fax: **+852-8203-2774**

Singapore Office
Axioma, (Asia) Pte Ltd.
30 Raffles Place
#23-00 Chevron House
Singapore 048622

Phone: **+65 6233 6835**
Fax: **+65 6233 6891**

Sales: sales@axiomainc.com
Client Support: support@axiomainc.com
Careers: careers@axiomainc.com